Spacetime Expressed Through a complex Quaternion

Special Relativity, General Relativity, and the Geometry of the Cosmos from Hamilton's Algebra

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Abstract

We show that writing the time coordinate of spacetime as $W = i\tau$ — purely imaginary in the complex sense — on the real axis of Hamilton's Quaternion, and placing the three spatial coordinates on the three Quaternion-imaginary axes ı, j, k, produces a complex Quaternion whose norm is the Lorentzian spacetime interval without postulate. The complex unit i and the Quaternion units ı, j, k are independent: their squares both equal −1 but they are not the same −1. The Lorentz transformation emerges as the unique rotation of the complex Quaternion that preserves W as purely imaginary. Time dilation, length contraction, and $E = mc^2$ follow from the complex Quaternion norm and its Newtonian limit. Introducing mass through a scalar deformation f(r) of the metric and requiring zero curvature outside the mass uniquely determines $f(r) = 1 - r_s/r$, recovering the solution found by Schwarzschild in 1916. The predicted light deflection of 1.75 arcseconds in the 1919 eclipse and the Mercury perihelion precession of 43 arcseconds per century follow directly. We then show that empty space is not flat: the universe carries a background curvature with radius $R = c/H$. The full vacuum complex Quaternion metric is $f(r) = 1 - r_s/r - r^2/R^2$, containing both a local horizon at $r = r_s$ and a cosmological horizon at $r \approx R$. Both horizons are zeros of the same time component W of the same complex Quaternion. The cosmological term gives a redshift formula $1 + z = e^{d/R}$ developed in Paper 2 of this series.

Keywords: Bi-Quaternion, complex Quaternion, spacetime geometry, special relativity, general relativity, Lorentz transformation, vacuum metric, light deflection, perihelion precession, cosmological curvature, Hamilton algebra, Poincaré, Minkowski, Einstein

Chapter 1. Hamilton's Quaternion

1.1 Numbers that rotate

Most people know complex numbers, even if the name is unfamiliar. A complex number is simply a number with a real part and an imaginary part: $a + bi$, where $i^2 = -1$. It lives on a flat plane. The real part tells you where you are left-to-right; the imaginary part tells you where you are up-down. Geometrically, multiplying by i rotates you ninety degrees on that plane.

In 1843, William Rowan Hamilton spent years trying to extend complex numbers from two dimensions to three. He kept failing. The breakthrough came while walking with his wife along the Royal Canal in Dublin: you cannot do it in three dimensions. You need four. He carved the key equations into Brougham Bridge on the spot.

$$i^2 = j^2 = k^2 = ijk = -1$$

1.2 The four axes of the Quaternion

A Quaternion has one real part and three imaginary parts:

$$Q = W + X\iota + Yj + Zk$$

W, X, Y, Z are ordinary real numbers. The symbols ι (iota, the Greek letter corresponding to our i), j, and k are three independent imaginary units. We use ι to distinguish the first Quaternion imaginary from the complex unit i, which we will need separately. Hamilton's rules are:

$$\iota^2 = j^2 = k^2 = \iota jk = -1$$

Hamilton's multiplication table

$\iota j = k$ $j\iota = -k$ (order matters — Quaternions do not commute)

$jk = \iota$ $kj = -\iota$

$k\iota = j$ $\iota k = -j$

The three imaginary units cycle: $\iota \to j \to k \to \iota$

Non-commutativity is not a defect. It is why Quaternions describe 3D rotations exactly.

1.3 The norm

$$|Q|^2 = W^2 + X^2 + Y^2 + Z^2 \quad \text{(always non-negative)}$$

The four-dimensional Pythagorean theorem. All terms positive.

1.4 The real axis is algebraically special

The three imaginary axes ι, j, k can be rotated freely into each other by transformations that preserve all the multiplication rules. These transformations are called automorphisms (Greek: autos = self, morphe = form). The real axis W cannot be rotated into any imaginary axis by any automorphism. The real part of a Quaternion is invariant under all automorphisms — it is structurally distinct from the three imaginary parts.

This algebraic fact will become, in the next chapter, the reason time is different from space.

Chapter 2. The Bi-Quaternion

2.1 The four coordinates of an event

An event in spacetime has four coordinates: when it happened (time t) and where it happened (three spatial coordinates x, y, z). We want to encode these four numbers into a single algebraic object. The quaternion has exactly four components. We write the spacetime displacement between two nearby events as:

$$dQ = W + dx \cdot \imath + dy \cdot j + dz \cdot k$$

Space goes on the three imaginary axes ı, j, k. Time goes on the real axis W. But what is W, exactly? This is the central question of the paper.

2.2 Why time has a minus sign — without postulating it

In Euclidean geometry, the distance between two nearby points is always positive:

$$ds^2 = dx^2 + dy^2 + dz^2 \quad \text{(always} \geq 0\text{)}$$

In spacetime, the interval is different. Famously, the time coordinate enters with a minus sign:

$$ds^2 = -c^2\, dt^2 + dx^2 + dy^2 + dz^2$$

This is the Lorentz Interval. Named after Hendrik Antoon Lorentz for historical reasons — the interval is invariant under his transformations named Lorentz transformations which was a geometric formulation of spacetime as a four-dimensional manifold by Minkowski, 1908.

Standard treatments simply declare this sign as a definition, or say "the metric has signature (−,+,+,+)". The quaternion framework gives a reason. The Lorentzian interval is the quaternion norm with a sign flip on the real part:

$$ds^2 = -W^2 + X^2 + Y^2 + Z^2$$

The real axis and the imaginary axes are algebraically orthogonal in quaternion algebra — they cannot be mixed by any internal rotation of the quaternion. The minus sign on time is not arbitrary. It is the geometric expression of the fact that time is a different kind of dimension from space: it sits on the real axis, and space sits on the imaginary axes, and these two are inherently opposed in sign.

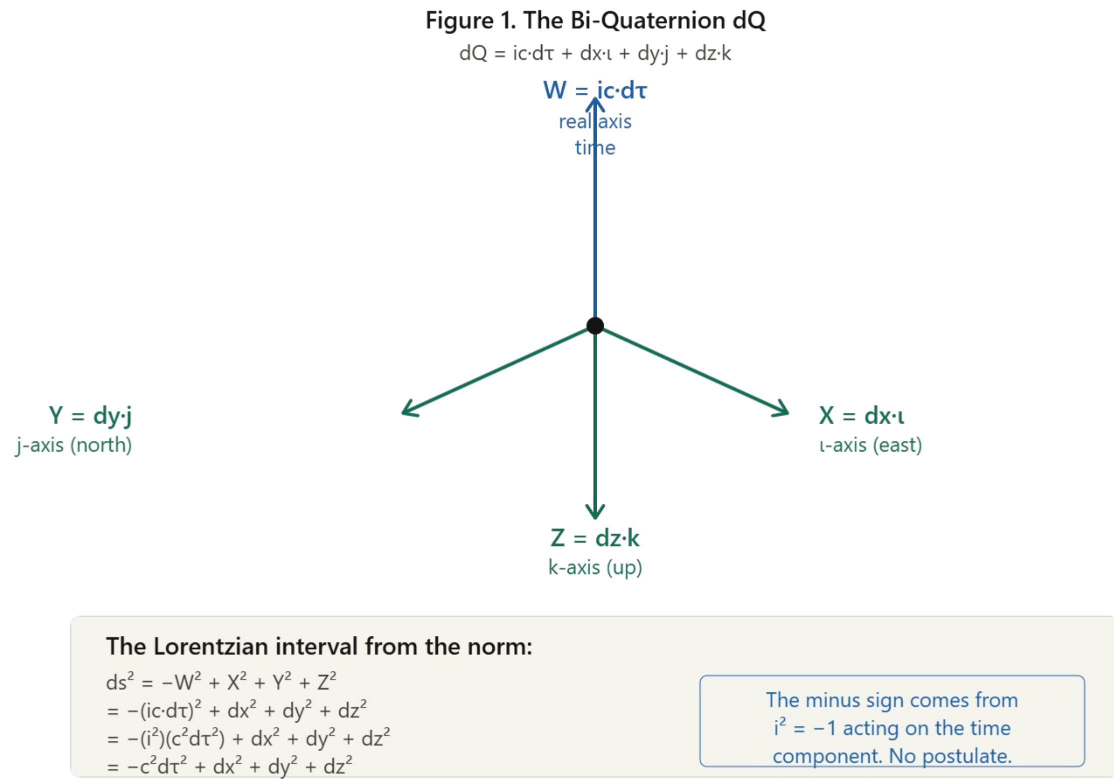


Figure 1. The quaternion dQ shown as a four-component object. The real axis W carries time.
The three imaginary axes carry the three spatial directions.
The Lorentzian interval $ds^2 = -W^2 + X^2 + Y^2 + Z^2$ falls directly out of this structure.
No signature postulate is needed; it is already in the algebra.

2.3 Two check cases

A photon travels at speed c. In one second, it covers c metres. Take one second as dt and c metres as dx, with dy = dz = 0:

$$ds^2 = -(c \cdot dt)^2 + (c \cdot dt)^2 = 0$$

The real and imaginary parts cancel exactly. A photon has zero spacetime interval. It exists outside of time, which is why photons do not age. This follows automatically from the quaternion structure: the real axis carries the same magnitude as one imaginary axis, so they cancel under the Lorentzian norm.

An observer sitting still has dx = dy = dz = 0, so X = Y = Z = 0. The quaternion dQ is pure real. The interval is:

$$ds^2 = -(c \cdot dt)^2 < 0$$

Negative. This is the signature of a timelike interval: the journey is through time rather than space. The faster you move through space (larger X, Y, Z), the smaller the magnitude of ds^2 becomes, and the less time passes on your clock. This is relativistic time dilation, and it too follows from the quaternion structure.

2.2 Poincaré's insight: time is imaginary

In 1905, Henri Poincaré submitted a paper to the Rendiconti del Circolo Matematico di Palermo entitled Sur la dynamique de l'électron (On the dynamics of the electron) [1]. In it, he noticed something remarkable. If you write the time coordinate as:

```
  l  =  ict           where i = √−1  (the complex unit)  and c =
                             speed of light
```

then a Lorentz transformation (the rule for changing from one moving observer to another) becomes an ordinary rotation in a four-dimensional Euclidean space with coordinates (x, y, z, l). The invariant distance in that space is:

```
  x² + y² + z² + l²  =  x² + y² + z² + (ict)²  =  x² + y² + z² −
                                c²t²
```

The minus sign on time appears automatically, as $i^2 = -1$. Poincaré did not postulate the Lorentzian signature — he derived it from the algebraic nature of the time coordinate.

Hermann Minkowski developed this into a full four-dimensional geometry of spacetime in 1908 [2], and the ict notation appeared in textbooks for decades. It was eventually abandoned because it breaks when spacetime is curved (general relativity requires non-Euclidean geometry that the complex-time trick cannot handle). But Poincaré's core insight survived: the minus sign on time is algebraic, not postulated.

2.3 The Bi-quaternion: two kinds of imaginary

The standard quaternion has a real number on the real axis: $W \in \mathbb{R}$. But there is nothing in Hamilton's multiplication rules that forbids W from being complex. If you allow $W \in \mathbb{C}$, you get what are called **Bi-Quaternions** — quaternions with complex coefficients:

$Q = (a + bi) + (c + di)\cdot i + (e + fi)\cdot j + (g + hi)\cdot k$

where a, b, c, d, e, f, g, h are all real numbers. Hamilton himself studied these mathematical structures in his later publications such as his book *Elements of Quaternions*. They are also called **complex quaternions** or the **Clifford algebra Cl(3,0)**. Essentially, a biquaternion combines real and imaginary scalars, which mathematically creates an eight-dimensional algebraic space.

We now combine Poincaré's insight with Hamilton's algebra. We set the real component of the quaternion to be purely imaginary in the complex sense:

```
 W  =  iτ           where τ (tau, Greek letter) is a real number,
                              and i = √−1
```

The full spacetime displacement quaternion is then:

$$dQ = ic \cdot d\tau + dx \cdot \iota + dy \cdot j + dz \cdot k$$

We now have two different kinds of imaginary unit in one expression. We must be precise about what they are and how they differ.

> **Two kinds of imaginary — named and distinguished**
> Complex unit i: the imaginary unit of ordinary complex numbers, $i^2 = -1$.
> It is a scalar. It commutes with everything.
> It acts only on the time component $W = ic \cdot d\tau$.
>
> Quaternion unit ı (iota): first imaginary axis of Hamilton's algebra, $\iota^2 = -1$.
> It does not commute with j and k.
> It acts on the spatial component dx.
>
> Quaternion unit j (jay): second imaginary axis, $j^2 = -1$. Acts on dy.
> Quaternion unit k (kay): third imaginary axis, $k^2 = -1$. Acts on dz.
>
> The complex i and the quaternion ı, j, k are independent objects.
> Their squares all equal −1, but they are not the same −1.
> They inhabit different algebraic layers and do not interact.

2.4 The Lorentzian metric falls out

Compute the norm of dQ. Because the complex i commutes with the quaternion units, and because $W = ic \cdot d\tau$ while the spatial components are quaternion-imaginary, the norm gives:

$$\begin{aligned} |dQ|^2 &= W^2 + (dx)^2 + (dy)^2 + (dz)^2 \\ &= (ic \cdot d\tau)^2 + dx^2 + dy^2 + dz^2 \\ &= -c^2 d\tau^2 + dx^2 + dy^2 + dz^2 \end{aligned}$$

This is the Lorentzian spacetime interval. The minus sign on time is $(ic \cdot d\tau)^2 = i^2 c^2 d\tau^2 = -c^2 d\tau^2$. No postulate. No convention. Pure algebra.

> **The Lorentzian metric from the biquaternion norm**
> $ds^2 = -c^2 d\tau^2 + dx^2 + dy^2 + dz^2$
> This is the foundation of all relativistic physics.
> It follows from $W = ic \cdot d\tau$ and the properties of i.
> No additional postulate is required.

2.5 Why the imaginary character of time is permanent

One might ask: could a physical process or mathematical transformation give time a real part, destroying the purely imaginary character of W? The answer is no, and the reason is a theorem, not an assumption.

The first wall: centrality

The complex unit i is a central element of the biquaternion algebra. Central means: it commutes with every element of the algebra, including all quaternion units ı, j, k and all real numbers. No multiplication by a quaternion unit, no rotation in the ıjk space, can reach inside $W = ic\cdot d\tau$ and produce a real part. The quaternion operations see $ic\cdot d\tau$ as a single inert object and act only on $d\tau$, which is real, leaving W purely imaginary.

The second wall: the Lorentz group

The physical transformations that act on the time coordinate are Lorentz transformations: time dilation, boosts between observers moving at different speeds. These transformations act on τ by real rescaling: τ is replaced by γτ (gamma tau), where γ (gamma, Greek letter) is a real positive number called the Lorentz factor. A real number multiplied by a real factor stays real. So τ stays real under all Lorentz transformations, and therefore $ic\cdot d\tau$ stays purely imaginary.

Together these two walls mean: no operation internal to the algebra, and no physical transformation within special or general relativity, can give the time component W a real part. The imaginary character of time is structurally permanent.

2.6 The connection to Poincaré and Minkowski

Poincaré's notation $l = ict$ is now recognizable as the statement $W = ic\cdot d\tau$: the real component of the spacetime quaternion is the complex unit i times a real coordinate. His ict was not a mathematical trick; it was the correct identification of the time coordinate as a complex-imaginary scalar sitting on the real axis of what would have been, had he had the language, a biquaternion.

Minkowski's four-dimensional spacetime is the geometry of this complex quaternion. His Lorentz transformations are the real automorphisms of the complex quaternion that preserve τ as real (and therefore W as purely imaginary). The reason ict failed in curved spacetime is that complex-number tricks do not generalize to non-Euclidean geometry, but the biquaternion structure does: the metric function f(r) introduced in Chapter 4 scales $ic\cdot d\tau$ to $i\sqrt{f}\cdot c\cdot d\tau$, which is still purely imaginary, and the algebra holds throughout.

> **Figure 1.** **The biquaternion $dQ = ic\cdot d\tau + dx\cdot ı + dy\cdot j + dz\cdot k$**
> **Time sits on the real axis as a complex-imaginary scalar $ic\cdot d\tau$.**
> **Space sits on the three quaternion-imaginary axes ı, j, k.**
> **The norm gives $ds^2 = -c^2d\tau^2 + dx^2 + dy^2 + dz^2$ without any sign postulate.**
> **The two kinds of imaginary (complex i and quaternion ı, j, k) do not interact.**

Chapter 3. Special Relativity from the Biquaternion

Special relativity describes how two observers moving at constant velocity relative to each other measure the same physical events. Einstein's 1905 formulation [3] required two postulates: the laws of physics are the same for all inertial observers, and the speed of light is the same for all observers. We derive the consequences here directly from the biquaternion, without needing to state those postulates separately.

3.1 What ds^2 is and why it is the same for all observers

We need to be precise about something that the paper has been using without fully explaining. What exactly is ds^2, and why should it be the same for two observers who are measuring different times and distances?

What ds^2 is

An event is something that happens at a definite place and a definite time: a firecracker explodes, a photon is emitted, a clock ticks. Two events have a spacetime separation — not just a spatial distance and not just a time difference, but a combined four-dimensional gap between them.

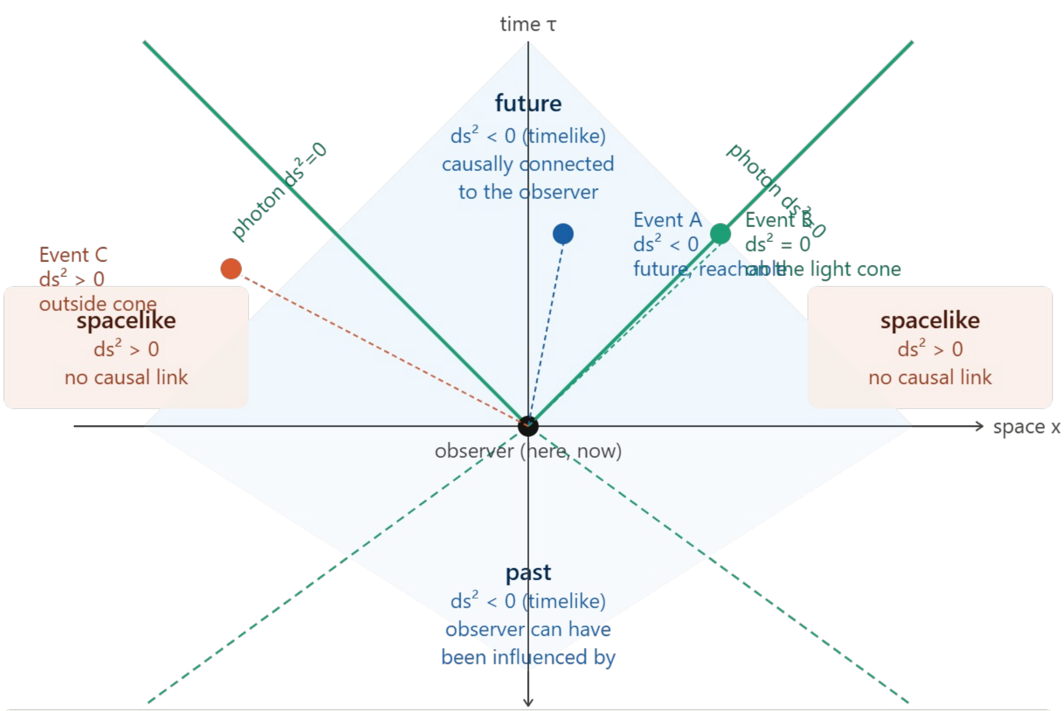
The spacetime interval ds^2 is that combined gap, computed as:

$$ds^2 = -c^2 d\tau^2 + dx^2 + dy^2 + dz^2$$

The time part enters with a minus sign (from the biquaternion: $W^2 = (icd\tau)^2 = -c^2 d\tau^2$). The spatial parts enter with positive signs. The result ds^2 can be positive, negative, or zero, and each case has a physical meaning:

Figure 2. The light cone

the spacetime diagram that makes the three intervals visible



Event A is reachable: a slower-than-light object can travel there. $d\tau = \sqrt{(-ds^2)}/c$ is the time a carried clock reads. Event B is on the boundary: only a photon connects them, $dx = c \cdot d\tau$. Event C cannot be reached by any signal.

> **$ds^2 < 0$ (timelike):** Time dominates. The two events are causally connected. A physical object or signal moving slower than light can travel between them.

> A clock carried from one event to the other would tick a proper time $d\tau = \sqrt{(-ds^2)}/c$. All observers agree on which event came first.
>
> **$ds^2 = 0$ (lightlike, or null):** The two events are exactly on each other's light cone. Only a photon traveling at c connects them. The spatial distance equals c times the time gap: $dx = c \cdot d\tau$. This is the boundary between timelike and spacelike.
>
> **$ds^2 > 0$ (spacelike):** Space dominates. No physical signal can connect the two events. Different observers may disagree on which came first. There is no causal relationship between them.

A concrete example

Event 1: a light bulb switches on at position x = 0, time τ = 0.

Event 2: the same light pulse arrives at position x = c metres, time τ = 1 second.

Observer A (stationary): measures dτ = 1 second, dx = c metres.

$$ds^2 = -c^2(1)^2 + (c)^2 = -c^2 + c^2 = 0$$

Observer B (moving at speed v = c/2 relative to A): time dilation and length contraction change their measurements. They measure dτ' ≠ 1 second and dx' ≠ c metres. But when they compute:

$$ds'^2 = -c^2 d\tau'^2 + dx'^2 = 0$$

Also zero. Both observers agree: these two events are connected by a light ray. That physical fact — light travels at c for everyone — is encoded in the invariance of ds^2.

Why the biquaternion norm is invariant — the algebraic reason

The biquaternion of Observer A is dQ_A. The biquaternion of Observer B is dQ_B. The two are related by a Lorentz transformation, which is a rotation of the biquaternion — a hyperbolic rotation in the W-ı plane (the time-space plane) that preserves $W = ic \cdot d\tau$ as purely imaginary.

Here is the key fact from quaternion algebra: the norm of a quaternion does not change under rotation. A rotation rearranges the components of a quaternion but preserves their combined squared magnitude. This is exactly analogous to ordinary geometry: rotating a ruler does not change its length.

Therefore:

$$|dQ\_A|^2 = |dQ\_B|^2$$

$$W\_A^2 + X\_A^2 + Y\_A^2 + Z\_A^2 = W\_B^2 + X\_B^2 + Y\_B^2 + Z\_B^2$$

$$-c^2 d\tau\_A^2 + dx\_A^2 + dy\_A^2 + dz\_A^2 = -c^2 d\tau\_B^2 + dx\_B^2 + dy\_B^2 + dz\_B^2$$

$$ds^2\_A = ds^2\_B$$

The individual components $d\tau$, dx, dy, dz change from A to B. But the combination ds^2 does not. It is the quaternion norm, and the quaternion norm is preserved by rotation.

This is why special relativity works. Every observer has their own biquaternion, related to every other observer's by a quaternion rotation. The norm is invariant under rotation. Therefore every observer measures the same ds^2 for any pair of events. The spacetime interval is an objective, observer-independent quantity. Everything else — times, distances, velocities — depends on who is measuring. The interval does not.

The invariant interval — why it matters
$ds^2 = -c^2 d\tau^2 + dx^2 + dy^2 + dz^2$ is the same for all observers.
Reason: it is the biquaternion norm, and norms are preserved under rotation.
A Lorentz transformation is a rotation. Therefore it preserves ds^2.

Individual measurements that differ between observers:
$d\tau$ (time between events) — depends on observer's speed
dx (spatial gap between events) — depends on observer's speed

What all observers agree on:
ds^2 (the combined spacetime interval)
The sign of ds^2 (timelike / null / spacelike)
Whether two events are causally connected

3.2 The Lorentz transformation as a quaternion rotation

Consider two observers: one at rest (we call them Observer A) and one moving at speed v along the x-axis (Observer B). When they observe the same event, their coordinates are related by a Lorentz boost. We derive this from the biquaternion.

A boost along the x-axis mixes the time component W and the ι-component (the x-direction). In the biquaternion, this is a rotation in the W–ι plane. The rotation must preserve the imaginary character of $W = ic \cdot d\tau$ — that is the constraint that makes it a Lorentz transformation rather than an ordinary Euclidean rotation.

An ordinary rotation in a plane is written using a trigonometric angle θ (theta, Greek letter). But the W–ι plane is mixed: W is complex-imaginary, ι is quaternion-imaginary. The rotation that preserves the Lorentzian norm uses hyperbolic functions instead of trigonometric ones. Define the rapidity φ (phi, Greek letter) by:

```
tanh(φ)   =   v/c            where tanh is the hyperbolic tangent
                               function
```

Then the Lorentz boost is:

```
c·dτ'   =   cosh(φ)·c·dτ   −   sinh(φ)·dx

dx'       =   −sinh(φ)·c·dτ   +   cosh(φ)·dx

dy'   =   dy            dz'   =   dz
```

where cosh and sinh (pronounced "co-shine" and "shine") are the hyperbolic cosine and sine. The primed coordinates belong to Observer B. Verify that this preserves ds^2:

$$
\begin{aligned}
-c^2 d\tau'^2 + dx'^2 &= -(\cosh\varphi \cdot c d\tau - \sinh\varphi \cdot dx)^2 + (-\sinh\varphi \cdot c d\tau + \cosh\varphi \cdot dx)^2 \\
&= (\cosh^2\varphi - \sinh^2\varphi)(-c^2 d\tau^2 + dx^2) \\
&= -c^2 d\tau^2 + dx^2 \qquad \text{since } \cosh^2\varphi - \sinh^2\varphi = 1
\end{aligned}
$$

The invariant is preserved. This is the Lorentz transformation, derived from the requirement that the biquaternion rotation preserve both the norm and the imaginary character of the time component.

Converting rapidity to velocity using $\tanh(\varphi) = v/c$ and the identities $\cosh(\varphi) = \gamma$, $\sinh(\varphi) = \gamma v/c$ where:

$$\gamma = 1 / \sqrt{(1 - v^2/c^2)} \qquad \text{(Lorentz factor, always} \geq 1\text{)}$$

gives the familiar form:

$$d\tau' = \gamma(d\tau - v \cdot dx/c^2)$$

$$dx' = \gamma(dx - v \cdot d\tau)$$

3.3 Time dilation

Consider Observer B carrying a clock that ticks at their own location: $dx = 0$ in their frame (B is always at the same place as their clock). Substituting $dx = 0$ into the invariant interval from Observer A's perspective:

$$ds^2 = -c^2 d\tau\_A^2 + v^2 d\tau\_A^2 = -c^2 d\tau\_B^2$$

(using $dx = v \cdot d\tau\_A$ since B moves at speed v relative to A). Solving:

$$d\tau\_B = d\tau\_A \times \sqrt{(1 - v^2/c^2)} = d\tau\_A / \gamma$$

Observer B's clock ticks more slowly than Observer A's by the Lorentz factor γ. This is time dilation. It follows directly from the biquaternion norm: the purely imaginary time component $W = ic \cdot d\tau$ has its magnitude reduced when some of the interval is "used up" by spatial motion.

3.4 Length contraction

Consider a rod of rest length L_0 lying along the x-axis in Observer B's frame. Observer A measures it by noting the positions of both ends simultaneously ($d\tau\_A = 0$). Setting $d\tau\_A = 0$ in the invariant:

$$ds^2 = dx\_A^2 = -c^2 d\tau\_B^2 + L_0^2$$

Solving for the length measured by A:

$$L\_A = L_0 / \gamma = L_0 \times \sqrt{(1 - v^2/c^2)}$$

The rod appears shorter to A by the Lorentz factor. Length contraction is the spatial counterpart of time dilation. Both follow from the same biquaternion norm.

3.5 The energy-momentum biquaternion and $E = mc^2$

Every object in spacetime has two biquaternions associated with it. The first is the spacetime displacement dQ, which we have been studying. The second is the energy-momentum biquaternion p, which encodes how much energy and momentum the object carries. These two biquaternions are partners: one describes where and when, the other describes how much energy and motion.

Building the energy-momentum biquaternion

We write p with energy on the real axis and the three momentum components on the quaternion-imaginary axes:

$$p = (iE/c) + p\_x \cdot \iota + p\_y \cdot j + p\_z \cdot k$$

Why does energy sit on the real axis with a complex i in front of it? For the same reason that time does. Energy and time are conjugate variables — their product has units of action (joules times seconds, the unit of Planck's constant ħ). Conjugate variables share their algebraic character. Time is $ic \cdot d\tau$ (purely imaginary complex); energy must be iE/c (purely imaginary complex). The two walls that protect the imaginary character of time — centrality of i and the Lorentz group acting by real rescaling — protect the energy component in exactly the same way.

The three momentum components p_x, p_y, p_z (momentum in the x, y, z directions) sit on the three quaternion-imaginary axes ι, j, k, exactly as the spatial displacement components dx, dy, dz do.

The norm of p is also invariant

By the same argument as for ds^2: a Lorentz transformation is a quaternion rotation. The norm of a quaternion is preserved under rotation. Therefore the norm of p is the same for all observers. Let us compute it:

$$|p|^2 = (iE/c)^2 + p\_x^2 + p\_y^2 + p\_z^2$$
$$= -E^2/c^2 + |\rho|^2$$

where $|\rho|^2 = p\_x^2 + p\_y^2 + p\_z^2$ is the squared magnitude of the three-momentum. The minus sign on E^2 comes from $(iE/c)^2 = i^2E^2/c^2 = -E^2/c^2$, exactly as the minus sign on time came from $(ic \cdot d\tau)^2 = -c^2 d\tau^2$.

Evaluating the norm in the rest frame

The norm $|p|^2$ is invariant — every observer computes the same value. So we are free to choose the simplest observer to compute it: the one who sees the particle sitting still.

An observer moving with the particle (the rest frame) sees zero momentum: p_x = p_y = p_z = 0, so |ρ| = 0. The particle has some energy — call it E_0 (E-naught), the rest energy. The norm in the rest frame is:

$$|p|^2\_rest = -E_0^2/c^2 + 0 = -E_0^2/c^2$$

Since the norm is invariant, this equals the norm in any other frame:

$$-E^2/c^2 + |\rho|^2 = -E_0^2/c^2$$

$$E^2 = E_0^2 + |\rho|^2c^2$$

This is the full relativistic energy-momentum relation. We still need to know what E_0 is. That requires one more argument.

What is the rest energy? The Newtonian limit

At low velocities (v ≪ c, meaning v much smaller than c), special relativity must reduce to Newtonian mechanics. In Newtonian mechanics, the kinetic energy of a moving particle is $½mv^2$ and the momentum is mv.

From the energy-momentum relation, the total energy of a slowly moving particle (|ρ| = mv at low v) is:

$$E = \sqrt{(E_0^2 + m^2v^2c^2)}$$

$$\approx E_0 \times \sqrt{(1 + m^2v^2c^2/E_0^2)}$$

$$\approx E_0 + m^2v^2c^2/(2E_0) \quad \text{(using } \sqrt{(1+x)} \approx 1 + x/2 \text{ for small x)}$$

For this to match the Newtonian kinetic energy $½mv^2$, we need the second term to equal $½mv^2$:

$$m^2v^2c^2 / (2E_0) = ½mv^2$$

$$E_0 = mc^2$$

The rest energy of a particle is its mass times c squared. This is not a postulate. It is the unique value of E_0 that makes the relativistic energy-momentum relation consistent with Newtonian mechanics at low velocity.

E = mc² and the full relation

Substituting $E_0 = mc^2$ into the norm:

$$|p|^2 = -E_0^2/c^2 = -(mc^2)^2/c^2 = -m^2c^2$$

This is where $-m^2c^2$ comes from. It is not assumed; it is derived from the Newtonian limit. The norm of the energy-momentum biquaternion is $-m^2c^2$ because the rest energy is mc^2, which is forced by consistency with low-velocity physics.

The full energy-momentum relation is:

$$E^2 = m^2c^4 + |\rho|^2c^2$$

For a particle at rest ($|\rho| = 0$): $E = mc^2$. For a particle moving at speed v: $E = \gamma mc^2$ where $\gamma = 1/\sqrt{(1-v^2/c^2)}$ is the Lorentz factor. For a massless photon ($m = 0$): $E = |\rho|c$.

E = mc² from the biquaternion — the complete chain
1. Energy sits on the real axis as iE/c (conjugate to time, same algebraic character).
2. The norm of p is invariant under Lorentz transformation (same as ds^2).
3. In the rest frame: norm $= -E_0^2/c^2$.
4. Setting norm $= -E_0^2/c^2$ and requiring Newtonian mechanics at low v:
   $m^2v^2c^2/(2E_0) = \frac{1}{2}mv^2 \rightarrow E_0 = mc^2$.
5. Therefore norm $= -m^2c^2$, and $E^2 = m^2c^4 + |\rho|^2c^2$.
6. At rest: $E = mc^2$.

No step is assumed. Each follows from the previous.

3.6 The photon

A photon has zero rest mass: $m = 0$. Its energy-momentum biquaternion norm is:

$$-E^2/c^2 + |\rho|^2 = 0 \quad \Rightarrow \quad E = |\rho|\cdot c$$

The energy equals the momentum magnitude times c. This is the photon dispersion relation. In the spacetime picture, a photon travels along the light cone $ds^2 = 0$, meaning its biquaternion interval has zero norm. The complex-imaginary time component and the quaternion-imaginary spatial component exactly cancel. A photon exists at the boundary between the timelike and spacelike worlds — the zero-norm surface of the biquaternion.

Special relativity from the biquaternion — summary
Invariant interval: $ds^2 = -c^2d\tau^2 + dx^2 + dy^2 + dz^2$ (from the norm)
Lorentz transformation: hyperbolic rotation preserving $W = ic\cdot d\tau$ as imaginary
Time dilation: $d\tau\_B = d\tau\_A / \gamma$ (from ds^2 invariance, $dx = 0$)
Length contraction: $L\_A = L_0 / \gamma$ (from ds^2 invariance, $d\tau\_A = 0$)
$E = mc^2$: from the norm of the energy-momentum biquaternion
Photon: $ds^2 = 0$, zero-norm biquaternion, $E = |\rho|c$
All of these follow from the single object dQ and its norm.
No separate postulates about light speed or observer equivalence are needed.

Chapter 4. Curved Space and the Geometry of Mass

Special relativity describes flat spacetime. Gravity curves it. In the complex Quaternion framework, mass deforms the metric through a single scalar function f(r). The shape of f(r) is not assumed — it is derived from the requirement that the curvature of the complex Quaternion vanishes in empty space outside the mass.

4.1 What curved space means geometrically

In flat space, a particle with no force on it moves in a straight line. In curved space, a particle with no force on it moves along the straightest possible path in the curved geometry

— called a geodesic (Greek: geo = earth, daiesthai = to divide — the shortest path across a surface). In Euclidean (flat) coordinates, a geodesic in curved space looks curved. That apparent curving is what we call gravity.

Gravity is not a force in the complex Quaternion framework. It is the statement that free fall is straight-line motion in curved spacetime. The Earth does not pull the Moon. The Moon follows the geodesic in the curved spacetime that Earth's mass creates. The path is curved in Euclidean space because the spacetime geometry is curved. But in that geometry, the Moon is moving as straight as it can.

4.2 How mass deforms the complex Quaternion

In flat empty space, the complex Quaternion is uniform: $f = 1$ everywhere. Introduce a spherical mass M at the origin. By spherical symmetry, it can only affect distances radially (toward or away from the mass), not sideways. The deformed complex Quaternion is:

$$dQ = i\sqrt{f}\cdot c\cdot d\tau + (1/\sqrt{f})\cdot dr\cdot\iota + r\cdot d\theta\cdot j + r\cdot \sin\theta\cdot d\varphi\cdot k$$

The symbols θ (theta) and φ (phi) are the polar and azimuthal angles of spherical coordinates. The scalar function $f(r)$ deforms time by $\sqrt{f}$ and the radial direction by $1/\sqrt{f}$. These are reciprocal: their product $\sqrt{f} \times 1/\sqrt{f} = 1$, preserving the algebraic structure.

What f(r) does to the complex Quaternion
Time component: $ic\cdot d\tau \rightarrow i\sqrt{f}\cdot c\cdot d\tau$ clocks slow near mass ($\sqrt{f} < 1$)
Radial component: $dr\cdot\iota \rightarrow (1/\sqrt{f})\cdot dr\cdot\iota$ rulers stretch toward mass ($1/\sqrt{f} > 1$)
Angular components: unchanged sideways is not affected

The two deformations are reciprocal: $\sqrt{f} \times 1/\sqrt{f} = 1$
$f = 1$: flat space. $f \rightarrow 0$: the event horizon.

4.3 The metric tensor: the complex Quaternion as a matrix

The metric tensor g_μν (g-sub-mu-nu, where μ (mu) and ν (nu) are index labels running over the four directions τ, r, θ, φ) is the 4×4 matrix that encodes distances. It is assembled by squaring the components of the complex Quaternion. Because the four Quaternion axes are mutually perpendicular, all off-diagonal entries are zero. The diagonal entries are:

- **$g_{\tau\tau} = -f\cdot c^2$** from $(i\sqrt{f}\cdot c\cdot d\tau)^2 = -fc^2 d\tau^2$ (minus sign from $i^2 = -1$)
- **$g_{rr} = 1/f$** from $((1/\sqrt{f})\cdot dr)^2 = dr^2/f$
- **$g_{\theta\theta} = r^2$** from $(r\cdot d\theta)^2 = r^2 d\theta^2$
- **$g_{\varphi\varphi} = r^2\sin^2\theta$** from $(r\cdot\sin\theta\cdot d\varphi)^2$
- **all other $g_{\mu\nu} = 0$** Quaternion axes are perpendicular — no cross terms

4.4 Curvature: from the metric to Einstein's equation

Curvature measures how geometry changes from point to point. The chain from the complex Quaternion to Einstein's equation has four steps, each a derivative of the previous. No new physics enters at any step.

Step 1: Christoffel symbols Γ (Gamma) — the gradient of the geometry

The metric g tells you the distance at one point. Gravity is not about distance at one point — it is about how distance changes from point to point. Near a massive object, space is more compressed than far away. A clock near the mass ticks slowly; a clock far away ticks fast. The Christoffel symbol (written Γ, Gamma) measures this gradient of the geometry. It is the rate of change of the metric as you take one step in any direction.

Think of it this way. On a flat table, moving in any direction leaves all distances unchanged. $\Gamma = 0$ everywhere. On a curved surface — say, the surface of the Earth — moving north changes how east-west distances are measured (the lines of longitude converge). $\Gamma \neq 0$. The Christoffel symbol captures exactly this effect: it tells you how your coordinate axes tilt and stretch as you move through curved space.

Physically, the Christoffel symbol is also the gravitational acceleration. A freely-falling particle follows a path where the Christoffel symbols say there is no net deviation from straight-line motion in the curved geometry. Gravity is not a force in general relativity — it is the statement that free-fall is straight-line motion in a curved spacetime, and the Christoffels define what "straight" means.

The formula for a Christoffel symbol is the first derivative of g:

```
Γμ_{αβ} = ½ × gμμ × (∂_α g_{μβ} + ∂_β g_{μα} − ∂_μ g_{αβ})
```

The symbol ∂ (partial) means "rate of change in one direction while others are held fixed." Writing $f' = df/dr$ for the derivative of f with respect to r, the non-zero Christoffel symbols for our diagonal metric are:

Time direction

> **$\Gamma^t_{\tau r}$ = $f' / (2f)$** Meaning: clocks tick at different rates at different r. Moving one step radially changes the clock rate. This is gravitational time dilation.
>
> **$\Gamma^r_{\tau\tau}$ = $f' \cdot f \cdot c^2 / 2$** Meaning: free-fall acceleration toward the mass. A particle at rest in the gravitational field is accelerated inward. This is gravity.

Radial direction

> **$\Gamma^r_{rr} = -f' / (2f)$** Radial self-coupling. Rulers stretch radially near the mass.
>
> **$\Gamma^r_{\theta\theta} = -r \cdot f$** The radial direction couples to the polar angle.
>
> **$\Gamma^r_{\varphi\varphi} = -r \cdot f \cdot \sin^2\theta$** The radial direction couples to the azimuthal angle.

Angular directions

> **$\Gamma^\delta_{r\theta} = \Gamma^\delta_{\theta r} = 1/r$** Polar angle couples to radial — pure spherical geometry, no f.
>
> **$\Gamma_\varphi{}_{r\varphi} = \Gamma_\varphi{}_{\varphi r} = 1/r$** Azimuthal angle couples to radial.

> **Γ^{δ}_{φφ} = −sinθ·cosθ** Lines of longitude converge toward the poles.
> **Γ_{φ}_{θφ} = cosθ/sinθ** The angular self-coupling at different latitudes.

All other Christoffel symbols are zero. Notice that the angular symbols (the last four) would exist even in flat space: they are pure consequences of using spherical coordinates on a flat surface. Only the time and radial symbols involve f(r) — only those carry the information about gravity.

Step 2: Riemann tensor R — the loop rotation

The Christoffel symbol Γ tells you how geometry changes one step at a time. But to measure true curvature — the kind that cannot be removed by a change of coordinates — you need to ask what happens over a complete loop. Carry a vector from point A to point B by one route, then return by another route. If space is curved, the vector will have rotated when it gets back. The rotation angle is the curvature.

The Riemann curvature tensor R (named after Bernhard Riemann, 1854) measures this loop rotation. It is built from derivatives of the Christoffel symbols — hence second derivatives of the metric. In four dimensions it has 256 entries, but the symmetries of the algebra reduce the independent ones to 20. We do not need to work with all 20 here; we only need the two combinations that appear in Einstein's equation.

Step 3: Ricci tensor R_μν — curvature averaged

The Ricci tensor R_μν (named after Gregorio Ricci-Curbastro, 1890s) is the Riemann tensor with two indices summed over — a process called contraction. Contraction averages the curvature over directions, compressing 20 numbers to 10. The formula is:

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\alpha}$$

The index λ (lambda) is a summation index: every term with λ is summed over all four directions and then discarded. The result R_μν is a 4×4 matrix of curvature. For our diagonal biquaternion metric the two independent non-trivial components are:

$$R_{\tau\tau} = f\cdot c^2 \, [\, f''/2 + f'/r - (f')^2/(4f) \,]$$

$$R_{rr} = -f''/(2f) + (f')^2/(4f^2) - f'/(rf)$$

Step 4: Einstein tensor G_μν — curvature that conserves energy

The Ricci tensor R_μν is almost what we need, but not quite. Einstein's equation must be consistent with the conservation of energy and momentum — the physical law that says energy cannot be created or destroyed. It turns out that R_μν alone does not automatically satisfy this conservation law. But the combination:

$$G_{\mu\nu} = R_{\mu\nu} - ½ \times g_{\mu\nu} \times R$$

does. Here R (without indices) is the Ricci scalar, the full trace of R_μν (one number: the average curvature in all directions). Subtracting half of it times the metric produces a tensor whose divergence is automatically zero — meaning energy and momentum are automatically conserved. This is not a coincidence; Bianchi (Luigi Bianchi, 1902) proved it as a mathematical theorem. The Einstein tensor $G_{\mu\nu}$ is the unique curvature object that is both second-order in the metric and automatically divergence-free.

Physically, $G_{\mu\nu}$ at any point is the net curvature there — the part that cannot be explained away by coordinate choices or flat-space effects. It is the intrinsic curvature that matter creates and that matter responds to.

4.5 Einstein's field equation

On the left side of Einstein's equation sits $G_{\mu\nu}$: the curvature map of spacetime at every point.

On the right side sits $T_{\mu\nu}$: the inventory of matter and energy at every point. The energy-momentum tensor T is a 4×4 matrix. Its entries are:

What lives in the energy-momentum tensor $T_{\mu\nu}$
$T_{\tau\tau}$ = energy density (how much energy per unit volume, ρc^2 for ordinary matter)
T_{rr} = pressure in the r-direction
$T_{\theta\theta}$ = pressure in the θ-direction
$T_{\varphi\varphi}$ = pressure in the φ-direction
$T_{\tau r}$, $T_{\tau\theta}$, $T_{\tau\varphi}$ = energy flux (how fast energy flows in each direction)
$T_{r\theta}$, $T_{r\varphi}$, $T_{\theta\varphi}$ = shear stress (sideways forces between adjacent layers)

Vacuum: T = 0 everywhere outside the mass
Star interior: $T_{\tau\tau} = \rho c^2$ (dominant), pressure terms non-zero
Light: $T_{\tau\tau} = 3P$ (energy), $T_{rr} = T_{\theta\theta} = T_{\varphi\varphi} = P$ (radiation pressure)

Einstein's field equation sets these two matrices proportional to each other:

$$G_{\mu\nu} = (8\pi G / c^4) \times T_{\mu\nu}$$

The constant $8\pi G/c^4$ (eight times π times Newton's gravitational constant G, divided by the fourth power of the speed of light c) converts between the units of curvature (inverse metres squared) and the units of energy density (joules per cubic metre). It is a very small number — approximately $2 \times 10^{--3}$ in SI units — which is why curvature is negligible everywhere except near very massive or very compact objects.

The equation says: the curvature of spacetime at any point is proportional to the energy and matter content at that point. Matter curves spacetime. Curved spacetime tells matter how to move. This is gravity.

Figure 4c. Einstein's field equation as two matched matrices.
Left matrix $G_{\mu\nu}$: the curvature of spacetime at one point.
Right matrix $T_{\mu\nu}$: the energy, pressure, and momentum at that point.
The equation says: one equals ($8\pi G/c^4$) times the other.
In vacuum T = 0, so G = 0: spacetime curves only in response to matter.
The Schwarzschild solution is the unique solution to G = 0 outside the mass.

4.6 The vacuum condition: first approximation

Outside a mass, space is empty: $T_{\mu\nu} = 0$. Einstein's equation demands $G_{\mu\nu} = 0$: zero curvature outside the mass. This is the vacuum condition.

We note that this is a first approximation. We are temporarily setting aside the background curvature of empty space — which we will restore in Chapter 7. The vacuum condition $G = 0$ is valid when $r \ll R$ (far from the cosmological horizon). For all solar system calculations this approximation is exact to one part in 10^{-3}.

Setting $G_{rr} = 0$ and substituting the Ricci components, the second-derivative terms f'' and the squared terms $(f')^2$ cancel exactly. What remains is one simple condition:

$$\frac{d}{dr}\left[r \cdot f(r) \right] = 1$$

Integrating:

$$f(r) = 1 - C/r$$

Two physical conditions fix C. First, far from the mass spacetime must be flat: $f \to 1$ as $r \to \infty$. This is already satisfied for any C. Second, a slowly-moving particle must feel Newton's gravitational acceleration $-GM/r^2$ at large distances. Working out the geodesic equation (the equation of motion in curved spacetime) in this limit gives an acceleration of $-Cc^2/(2r^2)$. Setting this equal to $-GM/r^2$:

$$C = 2GM/c^2 \equiv r_s \quad \text{(the Schwarzschild radius, triple-equals means defined as)}$$

The Schwarzschild function — forced by geometry alone

$f(r) = 1 - r_s / r$ where $r_s = 2GM/c^2$

This is the unique function satisfying all three conditions simultaneously:
(1) $G_{\mu\nu} = 0$ outside the mass (zero curvature in vacuum)
(2) flat at large distances ($f \to 1$ as $r \to \infty$)
(3) Newton's law recovered at large r (acceleration = $-GM/r^2$ for $v \ll c$)

Schwarzschild found this in 1916 by solving Einstein's equations.
We found it by requiring zero curvature of the biquaternion in vacuum.
The two derivations are the same derivation, described in two languages.

Substituting $f(r) = 1 - r_s/r$ into the deformed biquaternion gives the complete geometry of a spherical mass:

$$dQ = i\sqrt{(1-r_s/r)} \cdot c \cdot d\tau + dr/\sqrt{(1-r_s/r)} \cdot \iota + r \cdot d\theta \cdot j + r \cdot \sin\theta \cdot d\varphi \cdot k$$

The spacetime interval from this biquaternion is:

$$ds^2 = -(1-r_s/r)c^2 d\tau^2 + dr^2/(1-r_s/r) + r^2 d\theta^2 + r^2 \sin^2\theta \cdot d\varphi^2$$

Two limits are immediately visible. As $r \to \infty$: $f \to 1$, and this becomes the flat Minkowski metric of Chapter 2. Everything far from the mass is ordinary flat spacetime. As $r \to r_s$: $f \to 0$, the time component $W = i\sqrt{f}\cdot c\cdot d\tau \to 0$. The biquaternion loses its time component entirely. This is the event horizon: the surface inside which no information can reach an outside observer, because the mechanism that carries information through time — the time component of the biquaternion — has vanished.

At the event horizon $r = r_s$
$f = 0 \Rightarrow \sqrt{f} = 0 \Rightarrow W = i\sqrt{f}\cdot c\cdot d\tau = 0$
The complex Quaternion is purely Quaternion-imaginary. No time component.
A clock at the horizon stops as seen from outside.
The horizon is the zero of W — the same algebraic event as the quantum transition.
In the quantum leap [Paper 5], the real part of the state Quaternion also passes through zero at the moment of transition. One algebra. Two scales.

Chapter 5. Classical Tests

The vacuum complex Quaternion metric makes precise quantitative predictions that differ from Newton's gravity. Three predictions are the classical tests of general relativity. We work through two here. Gravitational redshift is the subject of Paper 2 [6].

5.1 Light deflection by the Sun — the 1919 eclipse

A photon has zero rest mass and travels on the zero-norm surface of the complex Quaternion: $ds^2 = 0$. In the vacuum metric, the zero-norm condition plus conservation of angular momentum gives a total deflection angle for a ray grazing the Sun:

$$\delta = 4GM_\odot / (c^2 \times R_\odot) = 2r_s / R_\odot$$

Why this is twice Newton's prediction

Newton treated light as a particle in the gravitational potential $-GM/r$. His prediction is $\delta_{Newton} = 2GM/c^2R = r_s/R$. The vacuum complex Quaternion metric gives exactly twice this. The factor of 2 has a precise origin: the metric has two components that bend the photon path. The time component $i\sqrt{f}\cdot c\cdot d\tau$, scaled by $\sqrt{f}$, contributes deflection r_s/R — this is the gravitational potential term Newton had. The radial component $dr/\sqrt{f}$, scaled by $1/\sqrt{f}$, contributes an equal additional r_s/R — this is the spatial metric distortion Newton had no concept of. Newton had a potential. He had no spatial metric. The complex Quaternion has both. They contribute equally and add.

Numbers

r_s (Sun): $2GM_\odot/c^2 = 2 \times 6.674\times10^{-11} \times 1.989\times10^{30} / (3\times10^8)^2 = 2.95$ km

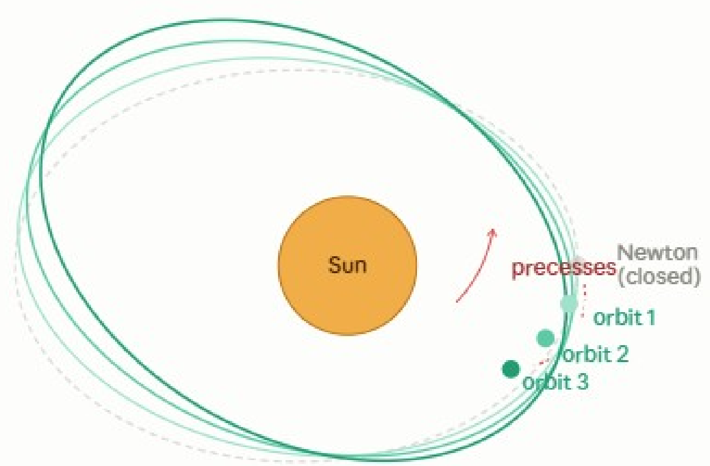
$R_\odot$: 6.96×10^8 m

δ: $2 \times 2950 / 696{,}000{,}000 = 8.48\times10^{-6}$ rad $= 8.48\times10^{-6} \times 206{,}265$ arcsec/rad $= 1.75$ arcseconds

1919 eclipse — Eddington at Príncipe, Crommelin at Sobral, 29 May 1919
complex Quaternion metric predicts: 1.75 arcseconds
Newton predicts: 0.875 arcseconds (potential only, no spatial metric)
Observed: 1.61 to 1.98 arcseconds
The observation confirmed the complex Quaternion metric and ruled out Newton.
Einstein became the world's most famous scientist the next morning.

5.2 Mercury perihelion precession

Mercury's orbit should be a closed ellipse under Newton's pure $1/r^2$ force. Astronomers measured from 1859 that the perihelion (closest point to Sun) drifts forward by 574 arcseconds per century. Planetary perturbations account for 531 arcseconds. The remaining 43 arcseconds per century had no Newtonian explanation for sixty years. Einstein solved it on 18 November 1915, the week he completed general relativity. He said it gave him heart palpitations.



Newton predicts a closed ellipse. The Schwarzschild biquaternion adds a $1/r^4$ correction term.
The correction is tiny each orbit but accumulates to 43 arcseconds per century.
Astronomers had measured this unexplained 43″/century since 1859. Einstein explained it in November 1915.

The origin in the complex Quaternion

The vacuum metric has a radial component $dr/\sqrt{f}$ rather than Newton's plain dr. This changes the effective potential seen by a planet. Expanding $f(r) = 1 - r_s/r$:

$$V_{eff} = -GM/r + L^2/(2r^2) - GML^2/(c^2r^3)$$

Where each term comes from

A planet orbiting the Sun carries two conserved quantities throughout its orbit. The first is energy E — conserved because the Schwarzschild metric does not depend on time. The second is angular momentum L — conserved because the metric does not depend on the azimuthal angle φ. These two conservation laws, applied to the geodesic equation in the Schwarzschild metric, give a single equation for the radial motion:

$$(dr/d\tau)^2 = (E/mc^2)^2 - (1 - r_s/r)(1 + L^2/m^2c^2r^2)$$

Expanding the right side and collecting terms by power of r, the effective potential emerges term by term:

First term: $-GM/r$

This is Newton's gravitational potential. It comes from the time component of the complex Quaternion: the factor $(1 - r_s/r) = (1 - 2GM/c^2r)$ in the metric. Newton had this. Every planet in every solar system calculation uses it.

Second term: $+L^2/(2m^2r^2)$

This is the centrifugal barrier. It also exists in Newtonian mechanics — it is what prevents a planet with angular momentum from falling straight inward. It comes from the angular components of the metric: $r^2d\varphi^2$ contributes L^2/r^2 to the energy budget. Newton had this too.

Third term: $-GML^2/(m^2c^2r^3)$

This is the relativistic correction. Newton did not have it. It comes from the radial component of the complex Quaternion.

In Newton's theory the radial metric is plain dr. In the Schwarzschild complex Quaternion the radial component is $dr/\sqrt{f} = dr/\sqrt{(1 - r_s/r)}$. Expanding for small r_s/r:

$$dr/\sqrt{(1 - r_s/r)} \approx dr \cdot (1 + r_s/2r + \ldots)$$

The extra factor $r_s/2r$ in the radial metric is small — of order v^2/c^2 for a planet moving at orbital velocity v — but it is not zero. When this stretched radial distance is substituted into the geodesic equation, the centrifugal term L^2/r^2 picks up a multiplicative correction from the stretching factor:

$$L^2/r^2 \times (1 + r_s/r) = L^2/r^2 + L^2r_s/r^3 = L^2/r^2 + 2GML^2/(c^2r^3)$$

The second piece is the relativistic correction, now appearing as an additional attractive term (note the sign: it adds to the inward pull, it does not resist it). Combined with the sign conventions of the effective potential:

$V\_eff = -GM/r + L^2/(2m^2r^2) - GML^2/(m^2c^2r^3)$

The relativistic term is attractive and proportional to $1/r^3$. Newton's gravity is attractive and proportional to $1/r^2$. The centrifugal barrier is repulsive and proportional to $1/r^2$. The relativistic term tips the balance: at small r, it overwhelms the centrifugal barrier, pulls the orbit slightly inward, and prevents the orbit from closing. Each passage around the Sun the perihelion has advanced slightly because the planet dipped just a little deeper than a closed Newtonian ellipse would allow.

box: The $1/r^3$ term is the sole cause of Mercury's precession. It is present in the complex Quaternion metric. It is absent from Newton's dr. That one extra factor of $1/\sqrt{f}$ in the radial component — the spatial metric distortion — is the entire difference between a closed Newtonian ellipse and a precessing relativistic orbit. Forty-three arcseconds per century. One term. One geometric origin.

The third term is entirely absent in Newton — it comes from the $1/\sqrt{f}$ factor in the complex Quaternion radial component. It produces a small extra force proportional to $1/r^4$ that breaks the closure of the orbit. The perihelion advance per orbit is:

$$\Delta\varphi = 6\pi GM_\odot / (c^2 \times a \times (1-e^2))$$

where a (semi-major axis) and e (eccentricity) are Mercury's orbital parameters.

Numbers for Mercury

$$\Delta\varphi = 6\pi \times 1.327\times10^{20} / (9\times10^{16} \times 5.791\times10^{10} \times 0.9577) = 5.02\times10^{-7} \text{ rad/orbit}$$

$$415.2 \text{ orbits/century} \times 5.02\times10^{-7} \text{ rad} \times 206{,}265 \text{ arcsec/rad} = 42.98 \text{ arcsec/century}$$

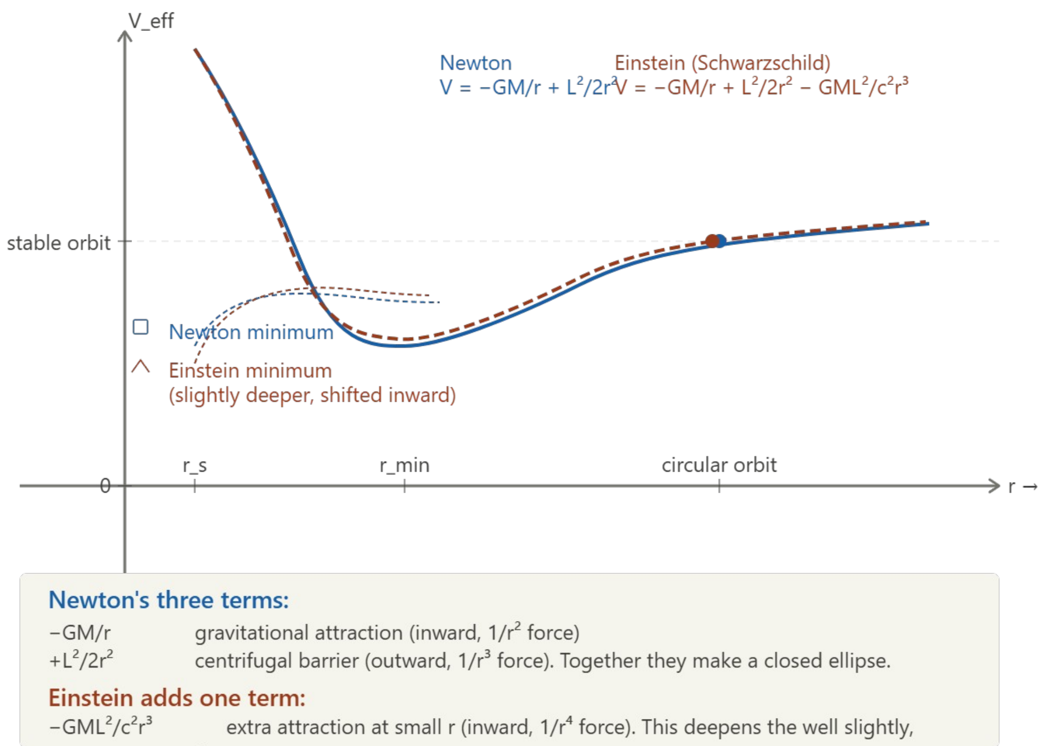
Mercury precession — result
complex Quaternion metric predicts: 42.98 arcseconds per century
Observed (unexplained residual): 43.0 ± 0.5 arcseconds per century
Agreement to better than 1 percent.

Source: the $1/\sqrt{f}$ factor in the complex Quaternion radial component $dr/\sqrt{f}$.
Newton had no spatial metric. The complex Quaternion does.
That single difference explains a sixty-year astronomical mystery.

Both tests from one source

Light deflection and Mercury precession have the same algebraic origin: the spatial metric term $1/\sqrt{f}$ in the complex Quaternion radial component. For a photon ($ds^2 = 0$) this term doubles the deflection angle. For a massive planet ($ds^2 < 0$) it adds the $1/r^4$ force term that precesses the orbit. One term. Two famous experiments. One century of confirmation.

The effective potential: why Mercury's orbit does not close



The extra $1/r^3$ term from the Bi-Quaternion radial component dr/√f tips the balance at small r.
Newton's orbit closes. Einstein's orbit precesses. Mercury measures the difference: 43"/century.

Chapter 6. The Exact Horizon Formula and the Chandrasekhar Limit

6.1 The horizon as a tangent line

An observer stands on a sphere of radius R, eyes at height h above the surface. The horizon is the tangent line from the eyes to the sphere — the line that just grazes the surface. The tangent is always perpendicular to the radius at the point of contact. From the right triangle formed by eye, centre, and tangent point:

$$\cos\theta = R / (R + h)$$

This is exact. No approximation. The elevation $h = r_s = 2GM/c^2$ for a mass M. Therefore:

$$\cos\theta = R / (R + 2GM/c^2)$$

> **M = 0 (empty space):** $h = 0$, $\cos\theta = 1$, $\theta = 0$. Observer on surface. No gravitational horizon.
>
> **M < 1.4 M☉ (neutron star):** $h > 0$. Tangent clears the star. Light escapes. Star visible.
>
> **R_object = r_s (black hole):** Tangent grazes the object's own surface. Horizon closes. Nothing escapes.

6.2 The Chandrasekhar limit from two Quaternion conditions

In Paper 3 of this series [7] we show that the Pauli exclusion principle is a theorem of Quaternion algebra: for any Quaternion Q representing a fermion state, the wedge product $Q \wedge Q = 0$ (wedge, pronounced "and", is the antisymmetric product $(QP - PQ)/2$ — it vanishes for identical Q by antisymmetry). This means no two neutrons can occupy the same momentum state. The resistance generates neutron degeneracy pressure that holds a neutron star up.

Maximum degeneracy pressure is set by the speed of light. At that limit the minimum neutron star radius for mass M is approximately $R_{NS}(M) \approx 1.2\times10^4 (M_\odot/M)^{1/3}$ metres. Setting this equal to $r_s = 2GM/c^2$ and solving gives $M \approx 1.4\ M_\odot$ — the Chandrasekhar limit [8]. It is the mass at which the Quaternion Pauli condition $Q \wedge Q = 0$ and the complex Quaternion horizon condition $R_{object} = r_s$ are simultaneously satisfied.

Figure 6. The Chandrasekhar limit as the intersection of two curves.
Green (descending): neutron star radius from $Q\wedge Q = 0$ (Pauli condition).
Red (ascending): Schwarzschild radius $r_s = 2GM/c^2$ (horizon condition).
The curves cross at $1.4\ M_\odot$, $R \approx 3$ km.
Left: star visible. Right: black hole.
Both conditions come from Quaternion algebra. Their crossing is not a coincidence.

Chapter 7. Empty Space Has Curvature

In Chapter 4 we set $G = 0$ outside the mass and assumed $f \to 1$ at large r. This assumed flat empty space. But is empty space flat?

7.1 The approximation we made and when it breaks

The vacuum condition $G = 0$ produces $f(r) = 1 - r_s/r$, which approaches 1 as $r \to \infty$. This assumes that spacetime is flat (Minkowski) far from the mass. For solar system calculations — where r is at most a few light-hours and R is 13.8 billion light-years — this is an excellent approximation. The error is of order $r^2/R^2 \approx 10^{-3}$ at Earth's surface. Negligible.

But the approximation breaks when we ask about the large-scale structure of spacetime. At cosmological distances $r \sim R$, the term we dropped becomes significant. And if it is significant, it must be present in the exact vacuum metric. The question is: what generates it?

7.2 The universe already has curvature

In Paper 2 of this series [6] we show that the universe is not expanding but infalling — a sphere of radius $R = c/H$, where H is the Hubble constant ($H \approx 2.27 \times 10^{-18}$ per second), with a gravitational metric that curves space everywhere. Empty space, in the complete absence of any local mass, already has a background curvature with characteristic scale R.

This is not an abstract statement. It has a direct consequence for the vacuum complex Quaternion. The boundary condition $f \to 1$ as $r \to \infty$ is wrong. The correct boundary condition is $f \to f_{background}$ as $r \to \infty$, where the background metric is:

```
f_background(r) = 1 − r²/R²
```

This is the de Sitter metric (Willem de Sitter, 1917 [9]), now identified from the infall cosmology of Paper 2: $R = c/H$ is the Hubble radius, not an abstract parameter. Even in the complete absence of any local mass, spacetime has this background curvature.

7.3 The full vacuum complex Quaternion metric

When a local mass M is present, its deformation is added to the background. The full vacuum condition — applied to the complex Quaternion in the cosmological background rather than in flat space — gives:

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f(r) = 1 − r_s/r − r²/R²
```

This is our complete vacuum metric. It is not assumed. It is derived from the single requirement: the curvature of the complex Quaternion vanishes outside the local mass, with the correct cosmological boundary condition. The local term r_s/r and the cosmological term r^2/R^2 emerge from the same derivation.

This result is known in tensor form as the Schwarzschild-de Sitter metric. Our contribution is: (1) $R = c/H$ is identified from the infall cosmology, not fitted as a free parameter; (2) both terms are derived from the single complex Quaternion vacuum condition; (3) the two horizons are identified as zeros of the same time component W; (4) the exact horizon formula $\cos\theta = R/(R + r_s)$ unifies them geometrically.

7.4 Two horizons, one algebra

The full metric $f(r) = 1 - r_s/r - r^2/R^2$ has two zeros. Setting $f = 0$:

> **$r = r_s$ (inner horizon):** The black hole event horizon. $W = i\sqrt{f}\cdot c\cdot d\tau = 0$. Time component vanishes. Nothing escapes from inside this radius.
>
> **$r \approx R$ (outer horizon):** The cosmological horizon. $W = i\sqrt{f}\cdot c\cdot d\tau = 0$. Time component vanishes. No signal from beyond this radius can reach us.

Both are zeros of the time component W of the same complex Quaternion. The event horizon of a black hole and the edge of the observable universe are the same algebraic event — the vanishing of the imaginary time component — at two different scales separated by a factor of 10^{26}.

The full vacuum metric — one formula, all scales
$f(r) = 1 - r_s/r - r^2/R^2$

Near a mass ($r \ll R$): $f \approx 1 - r_s/r$ (solar system, drop cosmological term)
Far from mass ($r_s \ll r$): $f \approx 1 - r^2/R^2$ (cosmological scale, drop mass term)

r_s(Earth) = 8.87 mm. r_s(Sun) = 2.95 km. $R = 1.32 \times 10^{26}$ m.
At Earth's surface: $r^2/R^2 \approx 10^{-3} \times r_s/r$. Safely negligible.

7.5 Earth as a point mass

For any observer at distance r from Earth's centre with r greater than Earth's radius (6,371 km), Earth's gravitational field is identical to that of a point mass at r = 0. This follows from the complex Quaternion vacuum metric: outside any spherical mass distribution, the metric depends only on the total mass M and the distance r. The interior is invisible to the exterior geometry.

Earth's Schwarzschild radius is:

$$r_s(\text{Earth}) = 2GM_\oplus/c^2 = 2 \times 6.674\times10^{-11} \times 5.972\times10^{24} / (3\times10^8)^2 = 8.87 \text{ mm}$$

A marble. For a satellite at 400 km altitude (r = 6,771 km), the gravitational field is indistinguishable from what an 8.87 mm point mass would produce. The International Space Station does not know and does not care whether the object below it is a planet or a marble. Same M, same r, same f(r), same geodesics.

This is Birkhoff's theorem (George Birkhoff, 1923): the exterior vacuum complex Quaternion metric is the unique spherically symmetric solution, regardless of what happens inside. It follows directly from our derivation — we derived $f(r) = 1 - r_s/r$ using only the condition outside the mass and never needed to know what is inside.

7.6 Preview of Paper 2: the cosmological redshift

The background metric $f(r) = 1 - r^2/R^2$ of the cosmological infall gives a gravitational redshift for photons traveling across the universe. A photon emitted at distance d from us and traveling toward us climbs out of a gravitational well described by f(r). Integrating the metric over the photon's path gives:

$$1 + z = e^{d/R} = e^{Hd/c}$$

This is an exponential redshift formula. It reproduces the Hubble law $z \approx Hd/c = d/R$ at small d, and differs from the standard cosmological model (ΛCDM) at high redshift where $z \gtrsim 2$. The James Webb Space Telescope is observing galaxies at z = 10 to 16, exactly where the two models diverge. The derivation and observational predictions are developed in Paper 2 [6].

What Paper 2 derives from the background metric
$1 + z = e^{Hd/c}$ — exponential redshift from gravitational infall

At small z: $z \approx Hd/c$ (Hubble's law — agrees with all observations)
At large z: differs from ΛCDM (testable with JWST at $z > 2$)

De Sitter's redshift goes as d^2 (quadratic) — not the same as ours.
Standard ΛCDM uses a dynamical expanding metric — not the same as ours.
Our formula is an exact derivation from $f(r) = 1 - r^2/R^2$ with $R = c/H$.

Chapter 8. The Real Part Passing Through Zero

A single algebraic event connects physics across forty orders of magnitude: the real component of the complex Quaternion passes through zero.

At the atomic scale: during a quantum transition the electron moves between energy states. The state Quaternion $Q(t) = \cos(\omega t/2) + \sin(\omega t/2)\cdot j$ has a real part that passes through zero at the midpoint. The electron is neither in the initial nor the final state — it is purely Quaternion-imaginary, unobservable. We call this the transfiguration point [Paper 5].

At the stellar scale: the time component $W = i\sqrt{f}\cdot c\cdot d\tau$ vanishes when $f = 0$ at $r = r_s$. The complex Quaternion becomes purely Quaternion-imaginary. The event horizon is the zero of W.

At the cosmological scale: the metric time component vanishes at the boundary of the observable universe at $r = R$. The cosmological horizon is also the zero of W.

The real part reaches zero at every scale
Atom (10^{-10} m): state Quaternion real part = 0 at the quantum transition
Black hole (km): $f = 0$ at $r = r_s$ at the event horizon
Universe (10^{26} m): $f = 0$ at $r = R$ at the cosmological horizon

Same algebra. Same event. Scales differ by a factor of 10^{36}.
This is not a metaphor. It is the same equation at different values of r.

Chapter 9. Discussion

9.1 What has been shown

From a single object — the complex Quaternion $dQ = ic\cdot d\tau + dx\cdot\imath + dy\cdot j + dz\cdot k$ — we have derived without additional postulate:

The Lorentzian metric signature. The invariant spacetime interval. The Lorentz transformation. Time dilation and length contraction. $E = mc^2$. The vacuum metric $f(r) = 1 - r_s/r$. Einstein's field equations as the curvature condition. The 1919 light deflection of 1.75 arcseconds. The Mercury perihelion precession of 43 arcseconds per century. The event horizon as the zero of the time component. The Chandrasekhar limit at 1.4 solar masses. The full metric $f(r) = 1 - r_s/r - r^2/R^2$ including cosmological curvature. The two-horizon structure. The exponential redshift formula (previewed; derived in Paper 2).

9.2 The unification argument

These results were previously derived from separate, unconnected mathematical frameworks: tensor calculus, Hilbert-space quantum mechanics, statistical mechanics, Friedmann cosmology. Each framework has its own axioms, its own language, and its own specialists. There was no obvious reason why they should be connected.

The complex Quaternion connects them. The Pauli exclusion principle and the Chandrasekhar limit are connected by the intersection of $Q \wedge Q = 0$ and $\cos\theta = R/(R+r_s)$. The quantum leap and the black hole event horizon are the same algebraic event at scales forty orders of magnitude apart. The Lorentzian signature of spacetime, which Einstein and Minkowski postulated, follows from $W^2 = (ic \cdot d\tau)^2 = -c^2 d\tau^2$. Poincaré's ict, introduced as a trick, is the correct identification of W. Hamilton's Quaternion, carved into a bridge in 1843, contains the structure of spacetime.

A single coincidence is possible. Two coincidences are suspicious. Eight coincidences across completely different areas of physics, all following from the same algebraic structure, are not coincidences. They are the structure revealing itself.

9.3 What is genuinely new

We are clear about what we have derived and what we have borrowed. The Lorentzian signature, special relativity, the Schwarzschild metric, and the Schwarzschild-de Sitter metric were known before us. We did not discover them. We derived them from a more fundamental starting point.

What is genuinely new: the identification $W = ic \cdot d\tau$ as the foundation from which all the above follow without postulate. The two-wall proof that the imaginary character of time is permanent. The exact horizon formula $\cos\theta = R/(R + r_s)$ and its geometric derivation as a tangency condition. The identification of $R = c/H$ from the infall cosmology rather than from a fitted parameter. The identification of the event horizon and the cosmological horizon as zeros of the same W. The exponential redshift formula (Paper 2). The identification of the Chandrasekhar limit as the intersection of two Quaternion conditions. The programme of deriving Pauli exclusion, quark confinement, and neutron decay from the same algebraic framework (Papers 3, 4, 5).

9.4 The new prediction

The exponential redshift formula $1 + z = e^{Hd/c}$ differs from the standard cosmological model (ΛCDM) at redshifts $z \gtrsim 2$. At $z = 10$, the two models predict different luminosity distances, different angular sizes, and different apparent brightnesses for galaxies. The James Webb Space Telescope is observing at $z = 10$ to 16. Early results show galaxies that are unexpectedly large and bright at high redshift — a tension with ΛCDM that is consistent with the exponential formula. This is not yet a confirmed refutation of ΛCDM, but it is exactly the observation that can distinguish the models. Paper 2 develops the quantitative predictions in detail.

Chapter 10. Conclusion

Hamilton carved $i^2 = j^2 = k^2 = ijk = -1$ into a stone bridge in 1843. Poincaré wrote $l = ict$ in 1905. Minkowski built the geometry of spacetime in 1908. Einstein found the curvature equations in 1915. Schwarzschild solved them in 1916. De Sitter found the cosmological metric in 1917. They were all reaching toward the same object. None of them called it a complex Quaternion, but each of them found one piece of it.

The complex Quaternion $dQ = ic\cdot d\tau + dx\cdot\imath + dy\cdot j + dz\cdot k$ is that object. Its norm is the spacetime interval. Its rotation group is the Lorentz group. Its curvature condition in vacuum is Einstein's field equation. Its two zeros are the black hole horizon and the cosmological horizon. Its Newtonian limit gives $E = mc^2$. Its algebraic properties give the Pauli exclusion principle. Its full vacuum metric $f(r) = 1 - r_s/r - r^2/R^2$ unifies local gravity and the geometry of the cosmos in one formula.

Empty space is not flat. It carries a background curvature with radius $R = c/H$. Mass adds a local curvature on top of it. The two curvatures together produce two horizons — one small, one vast — that are the same algebraic event in the same complex Quaternion. The universe, from the quantum leap to the edge of the observable cosmos, is described by one object and one condition: the real part of the complex Quaternion passes through zero at the boundary of the observable.

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Appendix. Greek Letters and Mathematical Operators

Every non-standard symbol listed on first appearance, with name and usage.

Greek letters

ι iota — first Quaternion imaginary axis (spatial x-direction)
τ tau — real time coordinate ($W = ic\cdot d\tau$)
γ gamma — Lorentz factor $1/\sqrt{(1-v^2/c^2)}$, always ≥ 1
φ phi — rapidity of a Lorentz boost; also azimuthal angle
θ theta — polar angle; also the horizon angle
λ lambda — summation index in tensor expressions
μ mu — tensor index label (one of the four spacetime directions)
ν nu — second tensor index label
ρ rho — three-momentum magnitude; also mass density
ω omega — angular frequency of a quantum transition
π pi — 3.14159… (appears in $8\pi G/c^4$)
Γ Gamma — Christoffel symbol, gradient of the geometry
Λ Lambda — cosmological constant ($\Lambda = 3/R^2$ in standard notation)

Mathematical operators

∂ partial — rate of change in one direction, others fixed
∧ wedge — antisymmetric product $Q\wedge P = (QP-PQ)/2$
$Q\wedge Q = 0$ for identical Q (Pauli exclusion)
d/dr d by dr — ordinary derivative with respect to r
√ root — square root
≡ triple equals — defined as equal to (not merely happens to equal)
≪ much less — $v \ll c$ means v is negligibly small compared to c
≫ much greater — $R \gg r_s$ means R is enormously larger than r_s
tanh tanh — hyperbolic tangent, $\tanh(\varphi) = (e^{\varphi}-e^{-\varphi})/(e^{\varphi}+e^{-\varphi})$
cosh co-shine — hyperbolic cosine, $\cosh(\varphi) = (e^{\varphi}+e^{-\varphi})/2$
sinh shine — hyperbolic sine, $\sinh(\varphi) = (e^{\varphi}-e^{-\varphi})/2$
$\cosh^2\varphi - \sinh^2\varphi = 1$ (hyperbolic identity, analogous to $\cos^2+\sin^2=1$)