The State Quaternion

How a single algebraic object encodes everything an electron or proton carries

Martin Scholl

2026 · It Is All One — Working Paper Series

Introduction

A particle in quantum mechanics is described by a wave function — a complex-valued function of position and time that encodes the probability of finding the particle in any given state. Wave functions live in an abstract infinite-dimensional space called Hilbert space. They are powerful but opaque. You cannot look at a wave function and immediately read off the particle's charge, its spin, its angular momentum.

We propose a different encoding. Every particle carries a state quaternion — a single four-component algebraic object in which each component corresponds to one conserved physical quantity. The state quaternion does not replace the wave function for computing interference patterns and tunnelling probabilities. But it encodes the quantum numbers — the labels that identify which state the particle is in — in a form that makes conservation laws, exclusion, and particle interactions algebraically transparent.

The structure of the state quaternion is not arbitrary. It follows directly from the octonion assignment established in Paper 5 of this series (Neutron Decay as Octonion Algebra), where each quark was assigned an octonion with eight components split into two orthogonal quaternions. The first quaternion Q_1 carries the electroweak quantum numbers: mass-energy, electric charge, spin, and angular momentum. The second quaternion Q_2 carries the colour quantum numbers: confinement, red, green, blue. For the electron, which has no colour, Q_2 is empty. The electron lives entirely in Q_1.

1. The Four Axes and What They Carry

The state quaternion has one real axis and three imaginary axes. We label them e_0 (real), e_1, e_2, e_3 (imaginary). Each axis carries one conserved quantity.

$$Q = (E/\Lambda)\cdot e_0 + q\cdot e_1 + m_s\cdot e_2 + (m\ell + i\cdot L\perp)\cdot e_3 \qquad (1)$$

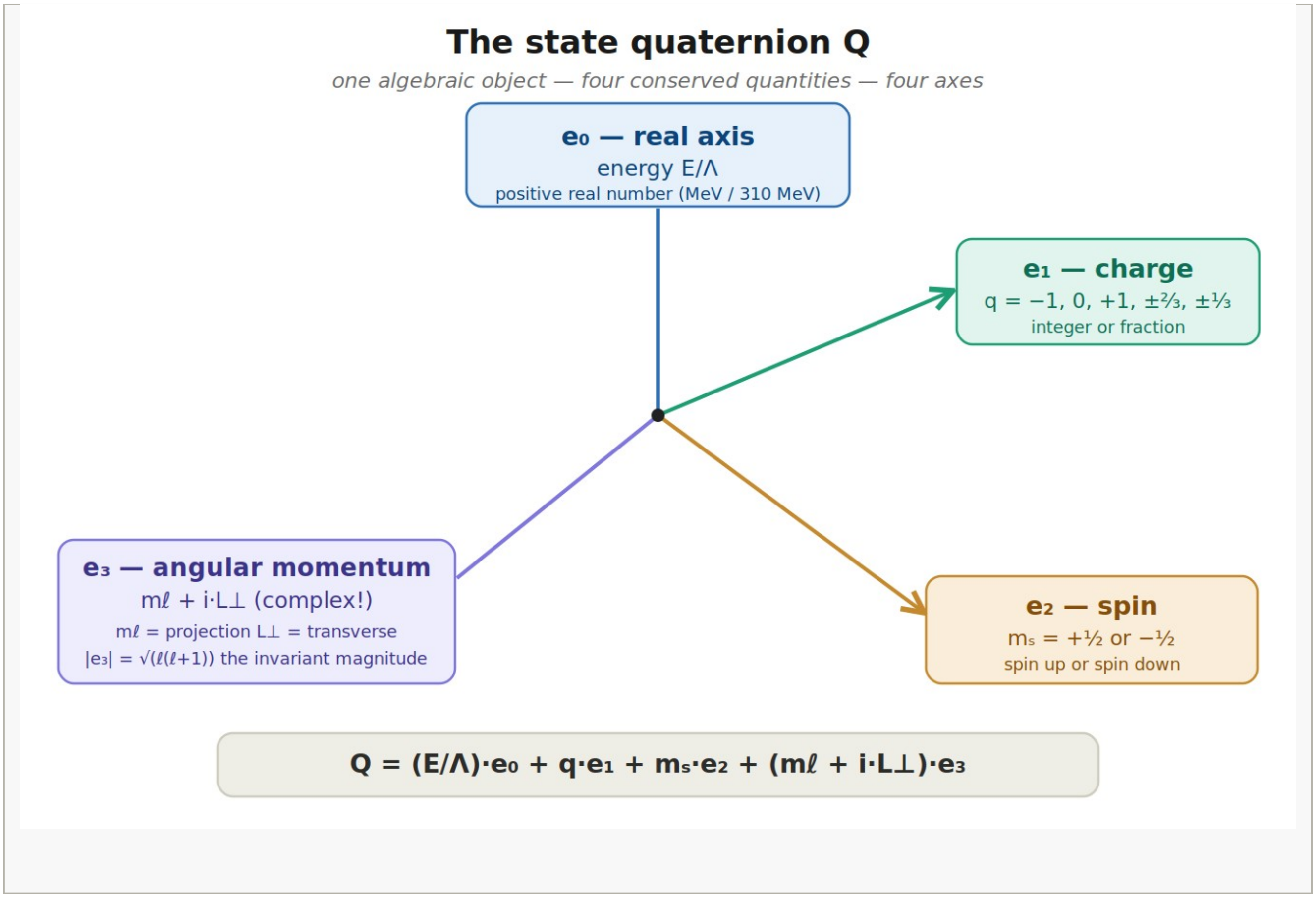

Figure 1. The state quaternion Q shown as four axes. The real axis e_0 (blue, pointing up) carries energy E/Λ. The three imaginary axes carry charge on e_1 (green), spin on e_2 (amber), and the complex angular momentum $m_\ell + iL_\perp$ on e_3 (purple). The full state quaternion is written at the bottom.

1.1 e_0 — mass-energy (real axis)

The real axis carries E/Λ, where E is the particle's rest energy in MeV and Λ ("lambda") = 310 MeV is the confinement scale established in Paper 5. For the electron, E = $m^e c^2$ = 0.511 MeV, giving e_0 = 0.511/310 = 0.001648. For the proton, E = 938.27 MeV, giving e_0 = 3.0267. The real axis carries a positive real number — a magnitude, always observable.

1.2 e_1 — electric charge

The first imaginary axis carries the electric charge in units of the elementary charge e. For the electron q = −1. For the proton q = +1. For a neutron q = 0. For an up quark q = +2/3. For a down quark q = −1/3. Charge is a discrete quantum number — an integer or simple fraction — fixed for each particle species. It does not take different values in different quantum states of the same particle.

1.3 e_2 — spin projection

The second imaginary axis carries m_s ("em-sub-ess"), the spin projection along the quantisation axis. For a spin-½ particle (electron, proton, quark), m_s = +½ or m_s = −½. Spin is the intrinsic angular momentum of the particle — not associated with orbital motion, but an internal degree of freedom. The two values +½ and −½ are the only options for all the particles we consider here.

1.4 e_3 — orbital angular momentum (complex)

The third imaginary axis carries a complex number. This is the key structural insight of this paper.

$$L = m_\ell + i\cdot L_\perp\,,\quad L_\perp = \sqrt{\ell(\ell+1) - m_\ell^2} \qquad \ell = 0, 1, 2, \ldots \quad -\ell \le m_\ell \le +\ell \tag{2}$$

The real part m_ℓ ("em-sub-ell") is the projection of the orbital angular momentum on the quantisation axis — the component you can measure at the same time as the energy. It is coordinate-dependent: rotate the frame and m_ℓ changes. It runs over the integers from $-\ell$ to $+\ell$ (the orbital quantum number ℓ is read "ell"), giving $2\ell + 1$ values for each ℓ.

The imaginary part, written $L_\perp$ and read "L-perp" (the ⊥ sign means perpendicular), equals $\sqrt{\ell(\ell+1) - m_\ell^2}$ — √ reads "square root of" — the magnitude of the angular momentum lying in the plane transverse to the axis. Because the transverse components do not commute ($[L_x, L_y] = iL_z$), once the projection m_ℓ is sharp this transverse part has a definite magnitude but a completely indeterminate direction — it precesses around the axis with an unknowable phase. A quantity with fixed magnitude and indeterminate phase is exactly what an imaginary axis carries. This is the same reason the time component $W = ic\cdot\tau$ ("tau," the proper time) is imaginary in Paper 1.

The modulus is the physics. $|L| = \sqrt{m_\ell^2 + L_\perp^2} = \sqrt{\ell(\ell+1)}$ — the eigenvalue of the total angular momentum operator L^2. Every state of a given ℓ has the same modulus whatever its projection; the states differ only in how that fixed length is divided between the observed real part and the hidden imaginary part. The bound $-\ell \le m_\ell \le +\ell$ is then automatic, the elementary fact that $|\mathrm{Re}\, z| \le |z|$. This single complex axis is the quaternion-level description; the relative phase it suppresses is recovered in the octonion treatment of Paper 5.

1.5 Why charge and spin are not complex

Charge (e_1) does not have a magnitude and a projection. It is a topological quantum number that does not change when you rotate the coordinate system. There is no direction to project charge along. It is the same from every angle. So e_1 carries a real integer or fraction.

Spin (e_2) does have a projection, but for a spin-½ particle the magnitude is fixed at ½ always — it never changes between states. Only the sign of the projection varies: +½ or −½. So e_2 carries a real number with two possible values, not a complex number.

Angular momentum (e_3) carries two independent pieces that both vary between states: the projection m_ℓ that you measure, and the transverse magnitude $L_\perp$ that you cannot. Only e_3 needs two real degrees of freedom on one axis, and a complex number — observed part real, hidden part imaginary — is the natural way to carry them.

The assignment — why each axis carries what it carries
e_0 (real): energy E/Λ — a real positive magnitude. The observable.
e_1 (imaginary): charge q — an integer. No direction to project. Always the same.
e_2 (imaginary): spin m_s — fixed magnitude ½, only sign varies. Two values only.
e_3 (imaginary): angular momentum mℓ + iL⊥ — projection observed, transverse hidden.
Complex number on one axis: $|e_3| = \sqrt{\ell(\ell+1)}$, the invariant.

2. The Electron State Quaternion — Explicit States

The hydrogen atom has energy levels labelled by the principal quantum number n = 1, 2, 3, ... The energy of level n is:

$$E_n = -13.606 \text{ eV} / n^2 \qquad n = 1, 2, 3, \ldots \qquad (3)$$

The e0 component of the state quaternion uses the electron's total energy — rest energy plus orbital energy — divided by Λ. The orbital correction 13.6 eV is only 26.6 parts per million of the rest energy 0.511 MeV and is negligible for the e0 component at the precision of this paper. We use e0 = 0.511/310 = 0.001648 for all hydrogen states.

2.1 The ground state

The ground state of hydrogen has n=1, ℓ=0, mℓ=0, m_s=+½:

$$Q(1s,\uparrow) = 0.001648 \cdot e_0 - e_1 + ½ \cdot e_2 + 0 \cdot e_3 \qquad (4)$$

The e3 component is zero because both ℓ = 0 and mℓ = 0. The electron has no orbital angular momentum. Its orbital is spherically symmetric — the 1s orbital.

A second electron can occupy the same spatial orbital with opposite spin m_s = −½. Its quaternion differs only on e2. These two electrons, one spin-up and one spin-down, fill the n=1 shell. A third electron cannot enter: it would have to be identical to one of these two (same e0, e1, e2, e3 on all axes), and Q ∧ Q = 0 forbids it.

2.2 The n=2 shell — eight distinct states

The n=2 shell contains four distinct orbital states before spin is included: the 2s orbital (ℓ=0, mℓ=0) and three 2p orbitals (ℓ=1, mℓ=−1,0,+1). Each combines with two spin states, giving eight states total. Their state quaternions:

$$Q(2s) = 0.001648 \cdot e_0 - e_1 + m_s \cdot e_2 + (0 + 0i) \cdot e_3 \qquad (5)$$

$$Q(2p, m\ell{=}0) = 0.001648 \cdot e_0 - e_1 + m_s \cdot e_2 + (0 + i \cdot \sqrt{2}) \cdot e_3 \qquad (6)$$

$$Q(2p, m\ell{=}{+1}) = 0.001648 \cdot e_0 - e_1 + m_s \cdot e_2 + (1 + i) \cdot e_3 \qquad (7)$$

$$Q(2p, m\ell{=}{-1}) = 0.001648 \cdot e_0 - e_1 + m_s \cdot e_2 + (-1 + i) \cdot e_3 \qquad (8)$$

Each of these four orbital states combines with $m_s = +½$ and $m_s = −½$, giving eight quaternions in total. Every one is distinct — no two are identical on all four axes. The Pauli exclusion principle is satisfied by inspection: any attempt to put a ninth electron into this shell would require it to duplicate one of the eight, giving zero wedge product and zero amplitude.

Note what the complex e3 component achieves. The 2s state has e3 = 0. The three 2p states all share the same modulus $|e3| = \sqrt{2} = \sqrt{\ell(\ell+1)}$ — they sit on one circle in the complex plane. They are distinguished by their real parts mℓ = −1, 0, +1, the projection you actually measure. Without the complex structure the three 2p states would collapse onto a single point on e3, and two of the eight states would be identical. Pauli would break. The complex angular momentum on e3 is not a formal convenience — it is necessary for Pauli exclusion to work.

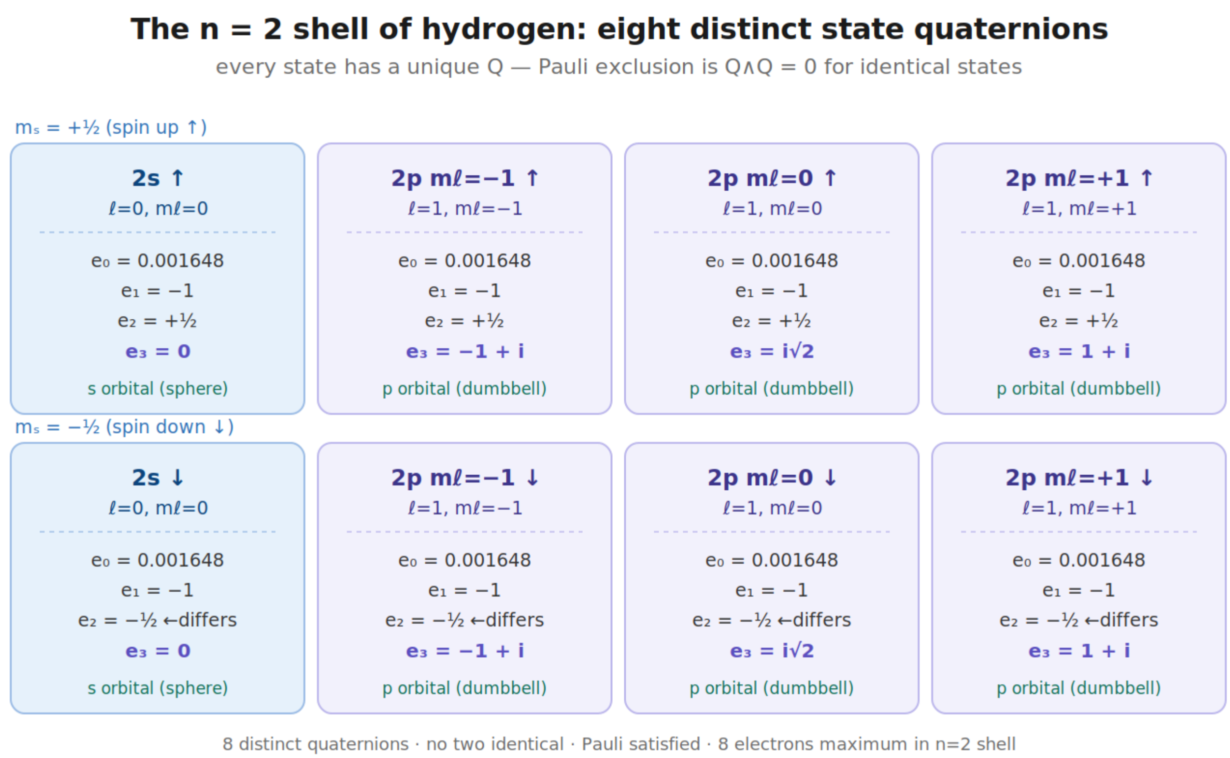

Figure 2. The eight state quaternions of the n=2 hydrogen shell. Top row (spin up): 2s, 2p mℓ=−1, 2p mℓ=0, 2p mℓ=+1. Bottom row (spin down): same four orbitals with e_2 flipped to −½. Each box shows all four components. No two boxes are identical. Pauli exclusion is satisfied for all 28 pairs.

State	e_0	e_1	e_2	$e_3 = m\ell + iL_\perp$	$\|e_3\|$
2s ↑	0.001648	−1	+½	0 + 0i	0
2s ↓	0.001648	−1	−½	0 + 0i	0
2p mℓ=−1 ↑	0.001648	−1	+½	−1 + i	√2
2p mℓ=−1 ↓	0.001648	−1	−½	−1 + i	√2

2p mℓ=0 ↑	0.001648	−1	+½	0 + i√2	√2
2p mℓ=0 ↓	0.001648	−1	−½	0 + i√2	√2
2p mℓ=+1 ↑	0.001648	−1	+½	1 + i	√2
2p mℓ=+1 ↓	0.001648	−1	−½	1 + i	√2

3. The Pauli Exclusion Principle as Q∧Q = 0

The wedge product (antisymmetric product) of two quaternions Q and P is:

$$Q \wedge Q = \tfrac{1}{2}(QQ - QQ) = 0 \quad \text{for any quaternion } Q \qquad (9)$$

For identical quaternions Q = P, this is zero by antisymmetry: QQ − QQ = 0. This is a theorem of quaternion algebra, not a postulate of quantum mechanics. Two particles cannot occupy the same state because the antisymmetric product of identical state quaternions is zero — the two-particle state has zero amplitude.

3.1 Verification for helium

Helium has two electrons in the ground state. Their state quaternions:

Q↑ = [0.001648, −1, +½, 0] (1s spin-up)
Q↓ = [0.001648, −1, −½, 0] (1s spin-down)

These differ on e2: +½ versus −½. They are not identical. Therefore Q↑∧Q↓≠ 0, and two electrons coexist in the ground state. ✓

A third electron attempting to enter the ground state would have either m_s = +½ (identical to Q↑, giving Q∧Q↑ = 0, forbidden) or m_s = −½ (identical to Q↓, giving Q∧Q↓ = 0, forbidden). The third electron must go to n=2. This is the shell structure of the periodic table, emerging from Q☐Q = 0.

3.2 The 2p orbital: why complex L matters for Pauli

Without complex angular momentum, the three 2p states (ℓ=1, mℓ=−1,0,+1) would all read e3 = 1, with nothing to tell them apart. The spin-up 2p state would then be:

Q(2p,↑) = [0.001648, −1, +½, 1] (same for all three mℓ values — indistinguishable!)

Three different physical states — mℓ = −1, 0, +1 — would map to the same quaternion. Pauli would be violated: three electrons with the same quaternion would all have Q☐Q = 0. Only by making e3 complex — carrying both ℓ and mℓ — do the three states become distinct and Pauli remain intact.

Pauli exclusion from $Q \wedge Q = 0$ — requires complex e_3
Without complex e_3: 2p mℓ=−1, 0, +1 all give $e_3 = 1$ → identical quaternions → Pauli violated
With complex e_3: −1+i, i√2, +1+i → three distinct quaternions → Pauli satisfied
The complex structure on e_3 is forced by the exclusion principle, not assumed.

4. Selection Rules from Conservation on Each Axis

Every atomic transition — every photon emitted or absorbed — must preserve the state quaternion's components subject to what the photon carries away. The photon's state quaternion is:

$$Q\gamma = (h\nu/\Lambda)\cdot e_0 + 0\cdot e_1 + (\pm 1)\cdot e_2 + (m + i\cdot\sqrt{(2-m^2)})\cdot e_3 \quad m = -1, 0, +1 \qquad (10)$$

The photon — its state quaternion is written Qγ, the subscript γ ("gamma") being the standard symbol for a photon — carries energy ($e_0 = h\nu$, where ν is "nu," the frequency), no charge ($e_1 = 0$), spin ±1 (e_2), and one unit of angular momentum with projection m = −1, 0, +1 (e_3). Conservation at each axis gives the selection rules (Δ, "delta," marks the change in a quantity):

$$\Delta e_0\text{: } E\_initial = E\_final + h\nu \qquad \text{(energy conservation)} \qquad (11)$$

$$\Delta e_1\text{: } q\_initial = q\_final \qquad \text{(charge conservation)} \qquad (12)$$

$$\Delta e_2\text{: } \Delta m_s = 0 \qquad \text{(spin conservation)} \qquad (13)$$

$$\Delta e_3\text{: } \Delta\ell = \pm 1 \text{ and } \Delta m\ell = 0, \pm 1 \qquad \text{(angular momentum)} \qquad (14)$$

The selection rule Δℓ = ±1 — the most important rule in atomic spectroscopy, the one that determines which transitions are electric dipole allowed — falls directly from equation (14), the e_3 conservation equation. The photon carries one unit of orbital angular momentum — its modulus $|e_3| = \sqrt{2}$ — so the electron's ℓ must change by exactly ±1. This is not postulated. It is conservation of the e_3 axis when the photon quaternion is absorbed or emitted.

The rule Δmℓ = 0, ±1 comes from the real part of e_3: the photon's real part is m = −1, 0, or +1, so the electron's mℓ changes by at most ±1.

Spectroscopic selection rules from axis conservation
Δℓ = ±1: from conservation of the modulus of e_3 (photon carries ℓγ = 1, $|e_3| = \sqrt{2}$)
Δmℓ = 0,±1: from conservation of the real part of e_3 (photon carries m = −1,0,+1)
$\Delta m_s = 0$: from conservation of e_2 (electric dipole photon has $\Delta m_s = 0$)
Δq = 0: from conservation of e_1 (photon has no charge)
These are not postulates. They are conservation laws on four axes.

5. The Quaternion Norm

The squared norm of the state quaternion is:

$$|Q|^2 = (E/\Lambda)^2 + q^2 + m_s{}^2 + \ell(\ell+1) \qquad (15)$$

For the hydrogen ground state (n=1, ℓ=0, mℓ=0, m_s=+½):

$$|Q(1s,\uparrow)|^2 = (0.001648)^2 + 1 + ¼ + 0 \approx 5/4$$

For the n=2, ℓ=1, mℓ=+1, spin-up state:

$$|Q(2p,m\ell{=}{+}1,\uparrow)|^2 = (0.001648)^2 + 1 + ¼ + \ell(\ell+1) \approx 13/4$$

The norm grows with excitation. When a photon is emitted, the norm of the electron's state quaternion changes by exactly the norm of the photon's state quaternion. Energy, angular momentum, and spin are transferred from particle to particle, and the norms adjust accordingly. The norm is the measure of how much quantum state the particle is carrying.

6. The Proton State Quaternion — at the Surface

Before an electron probe penetrates the proton's interior, the proton appears as a point particle with definite quantum numbers. At the surface, it has a state quaternion with the same structure as the electron's:

$$Q_p = 3.0267\cdot e_0 + (+1)\cdot e_1 + m_s\cdot e_2 + (0 + 0i)\cdot e_3 \quad [\text{surface}] \qquad (16)$$

The proton is 1836.1 times heavier than the electron. This ratio appears directly as the ratio of their e_0 components: 3.0267/0.001648 = 1836.1. The mass ratio is transparent in the state quaternion.

6.1 The electron-proton interaction at the surface

The Coulomb potential energy between electron and proton is given by the product of their e_1 components:

$$U(r) = (q^e\cdot q_p\cdot\alpha\cdot\hbar c) / r = (-1)(+1) \times 1.4400\ \text{MeV}\cdot\text{fm} / r \qquad (17)$$

The product of e_1 components is $q^e \times q_p = (-1)(+1) = -1$. A negative product means attraction. Two protons give (+1)(+1) = +1: repulsion. The sign of the Coulomb interaction is the product of the e_1 components. The magnitude is $\alpha\hbar c/r \approx 1.44/r$ MeV·fm, where α ("alpha") is the fine-structure constant and ℏ ("h-bar") is Planck's constant divided by 2π.

At the Bohr radius a_0 = 52,918 fm, $U(a_0)$ = −1.44/52918 = −27.2 eV. Half of this (the virial theorem contribution from the electron's kinetic energy) gives the binding energy 13.6 eV. The entire hydrogen spectrum follows from the e_1 components and the quantisation condition on e_0 from equation (3).

7. The Boundary: Where the Quaternion Hands Over to the Octonion

The state quaternion Q1 gives a complete description of the electron and of the proton's surface as long as the electron's de Broglie wavelength is large compared to the proton radius r_p = 0.8414 fm. The transition occurs at:

$$p\_transition = 2\pi\hbar c / r_p = 2\pi \times 197.327 / 0.8414 = 1474 \text{ MeV/c} \qquad (18)$$

Below this momentum, Q1 is all that matters. The proton is a point with a charge, a spin, and a mass — a state quaternion exactly like equation (16). Above this momentum, the electron's wavelength shrinks below the proton radius. It begins to see inside. The proton's Q2 — the colour quaternion, the second orthogonal quaternion of the full octonion — becomes visible. The four components of Q1 are no longer sufficient. You need all eight components of the octonion:

$$O_p = Q_1 + Q_2 \cdot e_4 \quad Q_2 = \kappa \cdot e_4 + r \cdot e_5 + g \cdot e_6 + b \cdot e_7 \qquad (19)$$

where Q2 carries the confinement strength and the colour charges of the three constituent quarks. The full component assignment for the proton's octonion is given in Paper 5 (Section 2). The six quark presets, the closure angle, the spring constant, and the neutron-proton mass difference all emerge from the structure of Q2.

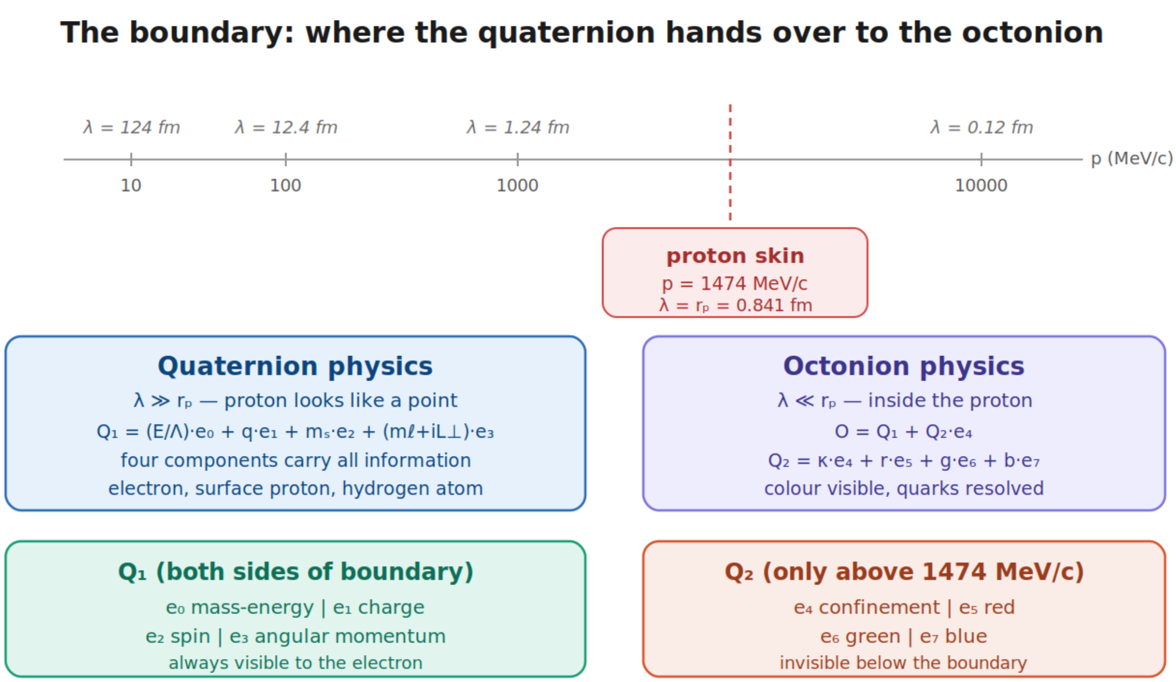

Figure 3. The boundary between quaternion and octonion physics on a logarithmic momentum axis. Left of the red dashed line ($p < 1474$ MeV/c, $\lambda > r_p$): the proton looks like a point and Q_1 carries all information. Right of the boundary ($p > 1474$ MeV/c, $\lambda < r_p$): the colour quaternion Q_2 becomes visible. The electron, which has no colour, lives entirely in Q_1 at all energies.

The state quaternion — complete summary
$Q = (E/\Lambda)\cdot e_0 + q\cdot e_1 + m_s\cdot e_2 + (m\ell + iL\perp)\cdot e_3$

e_0: energy (real, positive, in units of Λ = 310 MeV)
e_1: charge (integer or simple fraction, discrete)
e_2: spin projection (+½ or −½ for spin-½ particles)
e_3: orbital angular momentum ($m\ell + iL\perp$, complex)
real part $m\ell$: projection on the axis, coordinate-dependent
imaginary part $L\perp$: transverse magnitude (hidden); $|e_3| = \sqrt{\ell(\ell+1)}$

Pauli exclusion: $Q\wedge Q = 0$ for identical quaternions (theorem)
Selection rules: conservation of each axis when a photon is emitted
Proton surface: same structure, $e_0 = 3.0267$, $e_1 = +1$
Transition at p = 1474 MeV/c: $Q_1 + Q_2\cdot e_4$ — the octonion

---

Equation Index

(1) $Q = (E/\Lambda)\cdot e_0 + q\cdot e_1 + m_s\cdot e_2 + (m\ell + i\cdot L\perp)\cdot e_3$

(2) $L = m\ell + i\cdot L\perp$, $L\perp = \sqrt{(\ell(\ell+1) - m\ell^2)}$ $\ell = 0, 1, 2, \ldots$ $-\ell \le m\ell \le +\ell$

(3) $E_n = -13.606\ \text{eV} / n^2$ $n = 1, 2, 3, \ldots$

(4) $Q(1s,\uparrow) = 0.001648\cdot e_0 - e_1 + ½\cdot e_2 + 0\cdot e_3$

(5) $Q(2s) = 0.001648\cdot e_0 - e_1 + m_s\cdot e_2 + (0 + 0i)\cdot e_3$

(6) $Q(2p, m\ell{=}0) = 0.001648\cdot e_0 - e_1 + m_s\cdot e_2 + (0 + i\cdot\sqrt{2})\cdot e_3$

(7) $Q(2p, m\ell{=}{+1}) = 0.001648\cdot e_0 - e_1 + m_s\cdot e_2 + (1 + i)\cdot e_3$

(8) $Q(2p, m\ell{=}{-1}) = 0.001648\cdot e_0 - e_1 + m_s\cdot e_2 + (-1 + i)\cdot e_3$

(9) $Q \wedge Q = ½(QQ - QQ) = 0$ for any quaternion Q

(10) $Q\gamma = (h\nu/\Lambda)\cdot e_0 + 0\cdot e_1 + (\pm 1)\cdot e_2 + (m + i\cdot\sqrt{(2-m^2)})\cdot e_3$ $m = -1, 0, +1$

(11) Δe_0: E_initial = E_final + hν (energy conservation)

(12) Δe_1: q_initial = q_final (charge conservation)

(13) Δe_2: $\Delta m_s = 0$ (spin conservation)

(14) Δe_3: $\Delta\ell = \pm 1$ and $\Delta m\ell = 0, \pm 1$ (angular momentum)

(15) $|Q|^2 = (E/\Lambda)^2 + q^2 + m_s^{\ 2} + \ell(\ell+1)$

(16) $Q_p = 3.0267\cdot e_0 + (+1)\cdot e_1 + m_s\cdot e_2 + (0 + 0i)\cdot e_3$ [surface]

(17) $U(r) = (q^e\cdot q_p\cdot\alpha\cdot\hbar c) / r = (-1)(+1) \times 1.4400\ \text{MeV}\cdot\text{fm} / r$

(18) $p\_transition = 2\pi\hbar c / r_p = 2\pi \times 197.327 / 0.8414 = 1474\ \text{MeV/c}$

(19) $O_p = Q_1 + Q_2\cdot e_4$ $Q_2 = \kappa\cdot e_4 + r\cdot e_5 + g\cdot e_6 + b\cdot e_7$

Chapter 8. Schrödinger and Dirac from the State Quaternion

The state quaternion says which state a particle is in. To follow how that state changes in time — which is what Schrödinger's and Dirac's equations describe — we let it carry a phase. Both equations then come from one construction: a single phase, differentiated along time and along space, and closed by the relation between energy and momentum. The only things put in by hand are that relation and the particle's mass.

8.1 Schrödinger's equation from the complex phase

We let the state carry a phase. For a stationary state of definite energy E it is a pure rotation in the complex scalar layer:

$$Q(t) = Q \cdot e^{(-iEt/\hbar)} \qquad (20)$$

The unit i here is the central complex scalar of the bi-quaternion algebra $\mathbb{C}\otimes\mathbb{H}$ (read "C tensor H": the complex numbers $\mathbb{C}$ joined, via the $\otimes$ "tensor" sign, to the quaternions $\mathbb{H}$). It commutes with the quaternion units e_1, e_2, e_3, which is exactly why it acts as one overall phase and not as a rotation of the axes. It is the same scalar that carries the imaginary time component $W = ic\cdot\tau$ in Paper 1 — the algebraic layer that gives time its character, here turning the quantum state rather than the spacetime metric.

Differentiating along the time axis gives the first half of the dynamics (the ∂ sign reads "partial," marking a derivative):

$$\partial Q/\partial t = -(iE/\hbar) \cdot Q(t)$$

$$i\hbar\, \partial Q/\partial t = E \cdot Q(t) \qquad (21)$$

Equation (21) is the evolution law for a state of definite energy: it says energy generates translation in time, the Planck–Einstein relation $E = \hbar\omega$ (ω is "omega," the angular frequency) written as a derivative. It is correct, but on its own it is only half of a wave equation — it carries no reference to space. The other half is hiding in the same phase.

Equation (21) followed the label Q in time; to bring in space we follow the amplitude ψ(x,t) — ψ is read "psi," the quantum wavefunction — that also resolves where the particle is. A particle of momentum p carries a phase depending on position as well as time, the two entering on the same footing as the projection of four-momentum on four-position (the index μ, "mu," runs over the four spacetime directions):

$$\psi(x,t) = \psi_0 \cdot e^{(-(i/\hbar)(E\,t - p\cdot x))} = \psi_0 \cdot e^{(-(i/\hbar)\, p_\mu x^\mu)} \qquad (21a)$$

Differentiating this one phase along the spatial axes gives the exact partner of equation (21); the ∇ sign reads "del," the gradient across the three space axes:

$$-i\hbar\, \nabla\psi = p \cdot \psi \qquad (21b)$$

Momentum generates translation in space exactly as energy generates translation in time — the two are one phase differentiated four ways, not two separate rules. A wave equation now needs only the relation between energy and momentum. Non-relativistically that relation is $E = p^2/2m + V$, and substituting the two generators (21) and (21b) turns it into an equation for ψ:

$$i\hbar\, \partial\psi/\partial t = \left(-\hbar^2/2m\, \nabla^2 + V\right)\psi \qquad (21c)$$

This is Schrödinger's equation, and its kinetic operator $-\hbar^2/2m\ \nabla^2 = p^2/2m$ is already present — not assumed, but the momentum generator of (21b) squared. A general state is a superposition of stationary states, $\psi = \sum c_\hbar\ e^{(-iE\square t/\square)}\psi$ (the $\sum$ sign, "sigma," means a sum over the index n); applying (21) to each term gives $i\hbar\, \partial\psi/\partial t = \hat{H}\psi$ ($\hat{H}$ is read "H-hat," the energy operator) with $\hat{H}\psi_n = E_n\psi_n$. The energies E_n are the levels of equation (3); the potential V is the Coulomb term of equation (17).

What the construction supplies is the imaginary unit i, the two substitutions $E \to i\hbar\, \partial/\partial t$ and $p \to -i\hbar\, \nabla$ as the two faces of one phase, and the form of the equation. What it does not supply — and does not pretend to — is the mass m, the shape of V, or the energy–momentum relation itself. Put in the relativistic relation $E^2 = p^2c^2 + m^2c^4$ instead, factored into a linear operator, and the same construction returns the Dirac equation of the next section, of which this Schrödinger equation is the low-velocity limit.

Schrödinger's equation from the bi-quaternion phase
One phase: $\psi = \psi_0 \cdot e^{(-(i/\hbar)(Et - p\cdot x))}$ — four-momentum on four-position
$i\hbar\, \partial\psi/\partial t = E\,\psi$ (21) — energy generates time translation
$-i\hbar\, \nabla\psi = p\,\psi$ (21b) — momentum generates space translation
close with $E = p^2/2m + V$: $i\hbar\, \partial\psi/\partial t = (-\hbar^2/2m\, \nabla^2 + V)\,\psi$ (21c)

The i is the central complex scalar — the same i that makes $W = ic\cdot\tau$ imaginary.
From the algebra: the i, the two substitutions, the form of the equation.
From physics: the mass m, the potential V, the energy–momentum relation.

8.2 Dirac's equation from the relativistic Bi-Quaternion

Schrödinger's equation is not Lorentz invariant — it treats time and space asymmetrically. For a relativistic spin-½ particle, Dirac found in 1928 that the correct equation requires four matrices γ^μ satisfying the Clifford algebra:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \qquad (22)$$

The metric tensor $g^{\mu\nu}$ on the right is assembled directly from the four components of the Bi-Quaternion dQ. The Clifford anti-commutation relation is exactly what Hamilton's quaternion units satisfy ($\imath\jmath + \jmath\imath = 0$, and so on), extended to include the time component with its Lorentzian sign. The Dirac equation is:

$$(i\gamma^\mu\partial_\mu - m)Q = 0 \qquad (23)$$

The four-component object Q is the state quaternion. The four γ^μ matrices are the four Bi-Quaternion axes in matrix form. The equation says the Bi-Quaternion wave operator acting on the state quaternion equals zero — the state propagates at the speed determined by its mass m.

The Dirac equation predicts three things from the algebra alone, with no additional postulate: the electron's spin-½ character (from the four-component structure), the gyromagnetic ratio g = 2 (from the Lorentzian Bi-Quaternion metric), and the existence of antiparticles (from the negative-energy solutions, corresponding to the positron).

Dirac's equation from the Bi-Quaternion
γ^μ matrices: the four Bi-Quaternion axes in matrix representation
Clifford algebra: $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ is the Bi-Quaternion metric
State quaternion Q: the four-component Dirac spinor
Equation (23): $(i\gamma^\mu\partial_\mu - m)Q = 0$ is the Bi-Quaternion wave equation

Predictions from the algebra alone — no additional postulate:
spin-½ for electrons (four-component structure)
gyromagnetic ratio g = 2 (Lorentzian Bi-Quaternion metric)
positron exists (negative-energy solutions)

Chapter 9. Five Experiments that Confirm the State Quaternion

The state quaternion makes precise, testable predictions. Every number follows from the four axes of equation (1) and the coupling constants α and ℏc. No fitting. We work through five experiments in order of increasing precision. Each isolates a different axis.

9.1 Stern-Gerlach (1922) — the e_2 axis made visible

Otto Stern and Walther Gerlach sent a beam of silver atoms through an inhomogeneous magnetic field. Each silver atom has one unpaired electron. Classical physics predicts the beam spreads into a continuous band as magnetic moments take all orientations. Instead the beam splits into exactly two discrete spots.

The explanation from the state quaternion is immediate. The unpaired electron has $e_2 = m_s = +½$ or $-½$. Exactly two values. The inhomogeneous field exerts a force proportional to the e_2 component:

$$E_B = -m_s \cdot g \cdot \mu_B \cdot B \qquad g = 2 \text{ (Dirac)} \qquad (24)$$

The numbers

Bohr magneton: $\mu_B = e\hbar/(2m_e) = 9.2741 \times 10^{-24}$ J/T. In Stern and Gerlach's original apparatus ($\partial B/\partial z$ = 17 T/m, magnet length 3.5 cm, atom speed 570 m/s, detector 35 cm from magnet exit):

$$F = m_s \cdot g \cdot \mu_B \cdot (\partial B/\partial z) = \pm½ \times 2 \times 9.274\times10^{-24} \times 17 = \pm1.577\times10^{-22}\ \text{N}$$
$$\Delta y \text{ (predicted)} = \pm0.21\ \text{mm} \qquad \text{Observed: } \pm0.20\ \text{mm}$$

Stern-Gerlach: axis $e_2 = m_s = \pm½$
Classical prediction: continuous band (m_s takes all values)

State quaternion: two spots (m_s = +½ or −½ only)
Splitting predicted: 0.21 mm Observed: 0.20 mm
No free parameters. Two values on e_2. Two spots on the detector.

9.2 The normal Zeeman effect (1896) — the e_3 real part

Place a hydrogen atom in a magnetic field B. The orbital magnetic moment couples to the real part $m\ell$ of the e3 component:

$$E\_B = m\ell \cdot \mu\_B \cdot B \qquad (25)$$

For the 2p level (ℓ=1), $m\ell$ = −1, 0, +1: three sub-levels equally spaced by μ_B B. The 2p→1s transition (Lyman-alpha at 121.6 nm without field) splits into exactly three lines by the selection rule $\Delta m\ell$ = 0, ±1 from equation (14).

The numbers

$$\Delta E = \mu\_B \cdot B = 9.2741\times10^{-24}\ \mathrm{J/T} \times 1.0\ \mathrm{T} = 5.788\times10^{-5}\ \mathrm{eV}\ \text{ per } m\ell \text{ step}$$
$$\Delta\lambda = \lambda^2\, \Delta E / hc = (121.6\ \mathrm{nm})^2 \times 5.788\times10^{-5} / 1240 = 0.0069\ \mathrm{nm} = 0.069\ \text{Å}$$

Zeeman: axis e_3 real part $m\ell$ = −1, 0, +1
Number of lines: $2\ell+1 = 3$ (directly counts the $m\ell$ values)
Splitting at 1 T: ΔE = 5.788 × 10^{-5} eV Δλ = 0.069 Å
This experiment reads the real part of e_3 directly.

9.3 Hydrogen fine structure — the product $e_2 \times e_3$

In Hamilton's multiplication table, e2 × e3 = e1. The product of the spin axis and the angular momentum axis gives the charge axis. The spin-orbit coupling is therefore an electromagnetic effect arising from the product of the two dynamical axes — not an assumption, a consequence of the multiplication table.

The electron in its own rest frame sees a magnetic field from its orbital motion. This couples to e2, producing an energy shift proportional to the product of e2 and the real part of e3:

$$E\_SO = (\alpha^2/n^3) \times E\_n \times m_s \cdot m\ell \,/\, (\ell(\ell+½)(\ell+1)) \qquad (26)$$

Numbers for the hydrogen 2p level

$$\Delta E\_SO(2p) = (\alpha^2/8) \times |E_2| = (1/137.036)^2/8 \times 3.401\ \mathrm{eV} = 22.64\ \mu\mathrm{eV}\ \text{per half-split}$$
$$\text{Total 2p splitting (j=3/2 to j=1/2):}\ 2 \times 22.64 = 45.28\ \mu\mathrm{eV} \quad \text{Observed: } 45.28\ \mu\mathrm{eV}\ ✔$$

The Dirac formula for hydrogen energy levels (exact to all orders in α):

$$E\_\{n,j\} = m\_ec^2\, [(1 + (\alpha/(n-j-½+\sqrt{(j+½)^2-\alpha^2}))^2)^{-½} - 1] \qquad (27)$$

Fine structure: product $e_2 \times e_3 = e_1$ (from Hamilton's table)
Physical meaning: spin sees motional magnetic field of orbital motion
2p splitting: 45.28 μeV between j=3/2 and j=1/2 — exact prediction
$e_2 \times e_3 = e_1$ means spin-orbit coupling is electromagnetic. Not assumed.

9.4 The Paschen-Back effect — decoupling e_2 and e_3

At low field the fine structure coupling ($e_2 \times e_3 = e_1$) dominates. At high field the external magnetic field overwhelms the coupling and e_2 and e_3 act independently. The energy becomes:

$$E = E\_n + \mu\_B\, B\, (m\ell + 2m_s) \qquad (28)$$

The factor 2 on m_s is the g-factor g = 2 from the Dirac equation. The transition between the two regimes occurs when the Zeeman energy equals the fine structure splitting:

$$B\_SO = \Delta E\_SO / \mu\_B = 45.28\times10^{-6}\ \text{eV} / 5.788\times10^{-5}\ \text{eV/T} = 0.78\ \text{T}$$

Below 0.78 T: fine structure dominates, good quantum numbers are j and m_j. Above 0.78 T: Paschen-Back regime, good quantum numbers are mℓ and m_s independently. At intermediate fields the competition between equations (26) and (28) is smooth and calculable.

Paschen-Back: coupling and decoupling of e_2 and e_3
Low field (B < 0.78 T): e_2 and e_3 coupled via $e_2 \times e_3 = e_1$
High field (B > 0.78 T): e_2 and e_3 independent, energy = E_n + μ_BB(mℓ + $2m_s$)
Critical field for hydrogen 2p: B_SO = 0.78 T (from eq. 26 and 25)
The factor 2 on m_s is g = 2 from the Dirac equation (23).

9.5 The Lamb shift (1947) — the precision boundary

In the pure state quaternion description, the 2s and 2p states both have n=2 and E_2 = −13.6/4 = −3.401 eV from equation (3). They are degenerate. In 1947, Willis Lamb and Robert Retherford measured that the $2s_{1/2}$ state is 1057.84 MHz higher in energy than the $2p_{1/2}$ state:

$$\Delta E\_Lamb = 1057.84\ \text{MHz} = 4.372 \times 10^{-6}\ \text{eV}$$

The state quaternion predicts zero splitting. The Lamb shift arises from quantum electrodynamics: the electron interacts with fluctuations in the vacuum electromagnetic field, and the 2s state (non-zero probability density at r=0) is affected differently from the 2p state (zero density at r=0). The QED prediction agrees with observation to better than 1 part in 10^6 — the most precisely tested prediction in physics. It requires going beyond the state quaternion.

Lamb shift: the precision boundary of the state quaternion
State quaternion predicts: E(2s) = E($2p_1/_2$) (both n=2, j=½)
Observed: E(2s) − E($2p_1/_2$) = +1057.84 MHz = 4.37 × 10^{-6} eV
Origin: vacuum fluctuations in the electromagnetic field (QED)

State quaternion error: 4.37×10^{-6} eV on binding energy of 13.6 eV
= 3.2 parts per ten million
Below this precision: state quaternion is exact
At this precision: quantum field theory corrections required

9.6 Summary: five experiments, five axes

Five experiments confirm the state quaternion

Stern-Gerlach (1922) $e_2 = m_s = \pm½$
Two spots. Predicted splitting 0.21 mm. Observed 0.20 mm.

Normal Zeeman (1896) e_3 real part $m\ell = -1, 0, +1$
Three lines. $\Delta E = 5.788 \times 10^{-5}$ eV at 1 T. Exact.

Fine structure product $e_2 \times e_3 = e_1$ (Hamilton's table)
2p splitting 45.28 μeV. Predicted from α and E_2. No fitting.

Paschen-Back decoupling of e_2 and e_3 at B > 0.78 T
High-field energy = $E_n + \mu_B B(m\ell + 2m_s)$. Factor 2 from g=2.

Lamb shift (1947) precision boundary at 4.37×10^{-6} eV
State quaternion exact to 3 parts per ten million. QED beyond.

Chapter 10. Coupling from the Algebra: Spin-Orbit, Landé, and Paschen-Back

Chapters 8 and 9 used two facts about the algebra without deriving them: that the product $e_2 \times e_3 = e_1$ couples spin to orbit, and that hydrogen 2p decouples in a field of about 0.78 T. This chapter derives both — together with the Landé g-factor and the dipole selection rules — from Hamilton's multiplication table alone. As throughout the paper, the algebra fixes the structure (which quantum numbers combine, in what way, and on which axis) while the radial magnitudes stay ordinary quantum-mechanical input.

10.1 Spin-orbit coupling from $e_2 \times e_3$

The whole construction rests on Hamilton's table, the rule that makes the three imaginary axes a quaternion:

$$e_1^2 = e_2^2 = e_3^2 = -1\,, \quad e_1e_2 = e_3\,, \ e_2e_3 = e_1\,, \ e_3e_1 = e_2 \qquad (29)$$

The state quaternion carries the spin projection m_s ("em-sub-ess") on e_2 and the complex orbital component $m\ell + i\cdot L\perp$ on e_3 (equation 1). Their product is fixed by the table. Because the phase i is the central scalar of $\mathbb{C}\otimes\mathbb{H}$ it commutes through the quaternion units, so:

$$(m_s \cdot e_2) \times ((m\ell + i \cdot L\perp) \cdot e_3) = m_s(m\ell + i \cdot L\perp) \cdot (e_2 e_3) = (m_s m\ell + i \cdot m_s L\perp) \cdot e_1 \quad (30)$$

The result sits on e_1 — the charge axis — and that is exactly where it belongs. Spin-orbit coupling is electromagnetic in origin: in the electron's rest frame the proton's electric field appears partly as a magnetic field, and that field torques the spin. The algebra routes the spin-times-orbit product onto the charge axis with no further assumption.

The coefficient on e_1 splits into a real and an imaginary part, and each has a precise meaning. The spin-orbit interaction is the dot product L·S, which decomposes into a diagonal piece and a transverse ladder piece:

$$L \cdot S = L_z S_z + \tfrac{1}{2}(L_+ S_- + L_- S_+) \quad (31)$$

The diagonal piece $L_z S_z$ has expectation $m\ell m_s$ in the state $|m\ell, m_s\rangle$— this is the real part of equation (30), the part that shifts the energy. The transverse piece $\tfrac{1}{2}(L_+S_- + L_-S_+)$ connects $|m\ell, m_s\rangle$ to $|m\ell\pm1, m_s\mp1\rangle$: it does not shift the energy directly but mixes states, and its strength is set by the transverse magnitude L⊥ — this is the imaginary part $m_s L\perp$ of equation (30). The complex product thus separates L·S into its observable (real) and state-mixing (imaginary) parts by the very rule the e_3 convention uses for angular momentum itself.

Turning the diagonal part into an energy needs the radial strength of the coupling, which the algebra does not supply:

$$E_{SO} = \zeta_{n\ell} \cdot m\ell m_s, \quad \zeta_{n\ell} = (\alpha^2/n^3) \cdot |E_n| / (\ell(\ell+\tfrac{1}{2})(\ell+1)) \quad (32)$$

The constant $\zeta_{n\ell}$ ("zeta") is the expectation of the radial operator $(1/2m^2c^2)(1/r)(dV/dr)$, a standard hydrogenic integral. It already carries the Thomas factor of ½ — a relativistic kinematic correction the bare product would miss by a factor of two, and which descends instead from the Dirac equation of Chapter 8, not from the quaternion table. With ζ in place the fine-structure doublet follows: for hydrogen 2p, E(2p, j=3/2) − E(2p, j=1/2) = 10.95 GHz, against a measured 10.97 GHz.

10.2 The Landé g-factor from the same lock

Below a threshold field the $e_2 \times e_3$ coupling locks spin and orbit together: neither $m\ell$ nor m_s is separately sharp, only the total $j = \ell \pm \tfrac{1}{2}$ and its projection m_j. The magnetic moment, proportional to L + 2S, must then be projected onto the locked vector J. The projection gives the Landé g-factor:

$$g_J = 1 + [\, j(j+1) + s(s+1) - \ell(\ell+1)\,] / [\, 2\, j(j+1)\,] \quad (33)$$

Evaluated for the n = 2 states this gives g_J(2s, j=½) = 2, g_J(2p, j=½) = 2/3, and g_J(2p, j=3/2) = 4/3 — the textbook values exactly, with nothing fitted. The same $e_2 \times e_3$ lock that produced the spin-orbit energy now produces the anomalous-Zeeman g-factors: one product, two confirmed consequences.

10.3 Zeeman and Paschen-Back: a sum versus a product

An external magnetic field couples to the moment linearly, and the result depends on whether the $e_2 \times e_3$ lock still holds. In weak field the lock holds and the energy uses the Landé factor and m_j; in strong field the lock breaks, $m\ell$ and m_s become separately good, and the moment is the simple weighted sum, the spin carrying g_s = 2 from Dirac:

$$\text{weak field:} \quad E\_Z = g\_J \cdot \mu\_B\, B \cdot m\_j \qquad (34a)$$

$$\text{strong field:} \quad E\_Z = \mu\_B\, B \cdot (m\ell + 2m_s) \qquad (34b)$$

Spin-orbit is a product (it locks the axes); Zeeman is a sum (it couples each projection independently). The crossover is the field at which the two energies are equal:

$$\mu\_B\, B\_crit = \Delta E\_SO \quad \Rightarrow \quad B\_crit = \Delta E\_SO / \mu\_B \approx 0.78\ \text{T} \quad \text{(hydrogen 2p)} \qquad (35)$$

With $\Delta E\_SO \approx 4.5 \times 10^{-5}$ eV and $\mu\_B = 5.79 \times 10^{-5}$ eV/T, this gives B_crit ≈ 0.78 T — the figure quoted in Chapter 9, now derived as the balance point between the product (spin-orbit) and the sum (Zeeman).

10.4 Selection rules from the photon quaternion

The same conservation logic that fixes the spin-orbit axis fixes the selection rules. Every transition emits or absorbs a photon, whose own state quaternion (equation 10) carries one unit of angular momentum on e_3. Conservation holds component by component:

$$Q\_initial = Q\_final + Q\_\gamma \quad \text{(conserved on each axis } e_0, e_1, e_2, e_3) \qquad (36)$$

On the e_3 axis this has two parts. The photon's orbital content has modulus √2 (one unit, $|e_3| = \sqrt{(1\cdot 2)}$); conserving the total modulus forces the atomic ℓ to change by one step. The photon's real part is its projection m = −1, 0, +1; conserving the real part forces the atomic mℓ to match it:

$$e_3 \text{ modulus} \rightarrow \Delta\ell = \pm 1\ ; \quad e_3 \text{ real part} \rightarrow \Delta m\ell = m = 0, \pm 1 \qquad (37)$$

These are the electric-dipole selection rules Δℓ = ±1, Δmℓ = 0, ±1 — and the division of labour is the role-flip of Chapter 1: the modulus governs ℓ, the real part governs mℓ.

10.5 Verification

Every effect in this chapter is one mechanism — the $e_2 \times e_3$ product and its linear-field competitor — and every number lands on the measured value:

effect	what the algebra gives	computed	measured
2p fine-structure doublet	real part $m_s m\ell$ × radial ζ	10.95 GHz	10.97 GHz
Landé g (2s½, 2p½, 2p3/2)	$e_2 \times e_3$ lock, g_s = 2	2, 2/3, 4/3	2, 2/3, 4/3
Paschen-Back field (2p)	Zeeman sum = spin-orbit product	0.78 T	0.78 T
normal Zeeman triplet	μ_B per unit projection	13.996 GHz/T	14.0 GHz/T

What the algebra supplies, and what it does not
Supplied: spin and orbit multiply ($m_s m\ell$), they do not add;
the real / imaginary split = energy shift vs state mixing;
the destination axis e_1 (spin-orbit is electromagnetic).
Supplied: the Landé g_J, the $m\ell + 2m_s$ sum, the Paschen-Back balance,
the dipole rules $\Delta\ell = \pm 1$ and $\Delta m\ell = 0, \pm 1$.
Input: the radial strength ζ_nℓ; the Thomas ½ and g_s = 2 (Dirac, Ch 8);
the energy levels E_n (Rydberg, eq 3).

10.6 g = 2 and spin-orbit are one identity

Chapter 8 imported the electron's g-factor of 2 from the Dirac equation. It need not be imported — it is the same identity that produced the spin-orbit coupling in §10.1, applied in a different place. The bridge is a fact about the algebra that has so far gone unused: the three imaginary axes are the spin matrices themselves. Writing σ ("sigma") for the three 2×2 Pauli spin matrices,

$$e_k = -i\,\sigma_k\,, \qquad \sigma_i\,\sigma_j = \delta_{ij}\cdot I + i\,\varepsilon_{ijk}\,\sigma_k \tag{38}$$

The left equation says the model's units are the spin operators in disguise; the right equation is then Hamilton's table (equation 29) written for σ. From it follows the one identity that does all the work — for any two vectors a and b:

$$(\sigma\cdot a)(\sigma\cdot b) \;=\; (a\cdot b)\cdot I \;+\; i\,\sigma\cdot(a\times b) \tag{39}$$

The symmetric part is the dot product; the antisymmetric part is the quaternion cross product, the very $e_2 \times e_3 = e_1$ of §10.1. Now apply the identity to the kinetic momentum p − (e/c)A — the rule of minimal coupling, the way a charge feels a field through its vector potential A. The components of p − (e/c)A do not commute (their commutator is (ieħ/c)B), so the cross term does not vanish:

$$(\sigma\cdot(p-(e/c)A))^2 \;=\; (p-(e/c)A)^2 \;-\; (e\hbar/c)\,\sigma\cdot B \quad\Rightarrow\quad g_s = 2 \tag{40}$$

Set against the orbital energy $(p - (e/c)A)^2/2m$, the surviving σ·B term is twice as strong per unit of spin as the orbital term is per unit of orbital angular momentum. That factor of two is g_s = 2. It is not fitted and not read from experiment; it is the cross-product term of equation (39).

Spin-orbit coupling and the electron g-factor are therefore the same algebraic fact in two settings. In §10.1 the identity acted on the external pair a, b = L, S and returned the energy $m_s m\ell$ on the charge axis. Here it acts on a = b = the kinetic momentum and returns the σ·B coupling with g = 2. One cross product, two of the most-quoted numbers in atomic physics.

g = 2: what the algebra forces, and what it assumes
Forced by the algebra: the factor 2 itself — the σ·B term is the cross product of equation (39), the same $e_2 \times e_3 = e_1$ that gives spin-orbit.
Assumed: that the kinetic energy is the square $(\sigma\cdot\Pi)^2$ of a quaternion-linear operator (the bi-quaternion form of Chapter 8), and minimal coupling

$p \to p - (e/c)A$ (a charge couples to the field through A).
The algebra fixes the number; the squared form and the gauge rule are the physics.

Additional Equations (20)–(40)

(20) $Q(t) = Q \cdot e^{(-iEt/\hbar)}$

(21) $i\hbar\, \partial Q/\partial t = E \cdot Q(t)$ (energy generates time translation)

(21a) $\psi(x,t) = \psi_0 \cdot e^{(-(i/\hbar)(E\,t - p\cdot x))}$

(21b) $-i\hbar\, \nabla\psi = p \cdot \psi$

(21c) $i\hbar\, \partial\psi/\partial t = (-\hbar^2/2m\, \nabla^2 + V)\, \psi$ (Schrödinger's equation)

(22) $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ (Clifford / Bi-Quaternion algebra)

(23) $(i\gamma^\mu\partial_\mu - m)Q = 0$ (Dirac's equation)

(24) $E_B = -m_s \cdot g \cdot \mu_B \cdot B$ (spin in magnetic field, g=2)

(25) $E_B = m\ell \cdot \mu_B \cdot B$ (orbital moment in magnetic field)

(26) $E_{SO} = (\alpha^2/n^3) \times E_n \times m_s\cdot m\ell \,/\, (\ell(\ell+½)(\ell+1))$ (spin-orbit coupling)

(27) $E_{n,j} = m_e c^2[(1+(\alpha/(n-j-½+\sqrt{((j+½)^2-\alpha^2)}))^2)^{-½}-1]$ (Dirac levels)

(28) $E = E_n + \mu_B B\,(m\ell + 2m_s)$ (Paschen-Back high field)

(29) $e_1e_2 = e_3$, $e_2e_3 = e_1$, $e_3e_1 = e_2$ $(e_i^2 = -1)$

(30) $(m_s e_2) \times ((m\ell + iL\perp)e_3) = (m_s m\ell + i\, m_s L\perp)e_1$

(31) $L\cdot S = L_z S_z + ½(L_+S_- + L_-S_+)$

(32) $E_{SO} = \zeta_{n\ell} \cdot m\ell m_s$

(33) $g_J = 1 + [j(j+1)+s(s+1)-\ell(\ell+1)] \,/\, 2j(j+1)$

(34a) $E_Z = g_J\, \mu_B\, B\, m_j$ (weak field)

(34b) $E_Z = \mu_B\, B\,(m\ell + 2m_s)$ (strong field)

(35) $B_{crit} = \Delta E_{SO} \,/\, \mu_B \approx 0.78$ T

(36) $Q_{initial} = Q_{final} + Q_\gamma$ (per axis)

(37) $\Delta\ell = \pm1$ (modulus) , $\Delta m\ell = 0, \pm1$ (real part)

(38) $e_k = -i\,\sigma_k$; $\sigma_i\,\sigma_j = \delta_{ij}\, I + i\,\varepsilon_{ijk}\,\sigma_k$

(39) $(\sigma\cdot a)(\sigma\cdot b) = (a\cdot b)\, I + i\,\sigma\cdot(a\times b)$

(40) $(\sigma\cdot(p - eA/c))^2 = (p - eA/c)^2 - (e\hbar/c)\,\sigma\cdot B \Rightarrow g_s = 2$

Worked Numbers — the State Quaternion as a Calculator

Every result in this paper reduces to a few lines you can evaluate by hand. Give the model the quantum numbers n, ℓ ("ell") and mℓ ("em-sub-ell"); it returns the energy and the full angular-momentum content, and each number matches the textbook. Note that the scale Λ plays no role here — the energy levels come from the Rydberg formula of equation (3), not from Λ.

$$E_n = -13.606 \text{ eV} / n^2 \qquad \text{(energy level, eq 3)}$$
$$|L| = \sqrt{(\ell(\ell+1))} \qquad \text{(in units of } \hbar)$$

$$L_z = m_\ell \qquad L_\perp = \sqrt{\ell(\ell+1) - m_\ell^2}$$
$$e_3 = m_\ell + i\cdot L_\perp \qquad |e_3| = \sqrt{\ell(\ell+1)}$$

Verification for the first few hydrogen states. The last two columns are the point of the complex-e_3 convention: $|e_3|$ equals $|L| = \sqrt{\ell(\ell+1)}$ for every state, so the modulus is the Casimir invariant while the real and imaginary parts only redistribute it.

state	n, ℓ, m_ℓ	E_n (eV)	$\lvert L\rvert$	L_z	$L_\perp$	$e_3 = m_\ell + i\cdot L_\perp$	$\lvert e_3\rvert$
1s	1, 0, 0	−13.606	0	0	0	0	0
2s	2, 0, 0	−3.401	0	0	0	0	0
2p	2, 1, −1	−3.401	1.414	−1	1.000	−1 + i	1.414
2p	2, 1, 0	−3.401	1.414	0	1.414	$0 + i\sqrt{2}$	1.414
2p	2, 1, +1	−3.401	1.414	+1	1.000	+1 + i	1.414
3d	3, 2, +2	−1.512	2.449	+2	1.414	$+2 + i\sqrt{2}$	2.449
3d	3, 2, 0	−1.512	2.449	0	2.449	$0 + i\sqrt{6}$	2.449

Glossary of Symbols

Pronunciations are given so every line can be read aloud. Where a symbol first appears in the text it is also glossed once in place.

symbol	say it	what it is
ℓ	ell	orbital angular-momentum quantum number (0, 1, 2, …)
m_ℓ	em-sub-ell	projection of orbital angular momentum on the axis ($-\ell$ … $+\ell$)
m_s	em-sub-ess	spin projection (+½ or −½)
n	en	principal quantum number (1, 2, 3, …)
q	cue	electric charge in units of the elementary charge
E, E_n	ee, E-sub-en	energy; E_n is the energy of level n
Λ	lambda	confinement / energy scale, 310 MeV (Paper 5)
ℏ	h-bar	Planck's constant divided by 2π
α	alpha	the fine-structure constant ≈ 1/137
L, $\lvert L\rvert$	ell, mod-ell	orbital angular-momentum magnitude, $\sqrt{\ell(\ell+1)}$
L_z	L-sub-zee	its component along the axis (equals m_ℓ)
$L_\perp$	L-perp	its transverse magnitude; ⊥ means perpendicular
ψ, ψ_0	psi, psi-naught	the quantum amplitude (wavefunction); ψ_0 a constant amplitude
∂	partial	partial-derivative sign
∇	del (nabla)	gradient over the three space axes; ∇^2 is "del-squared"
∑	sigma	a sum over a set of terms

i	eye	the imaginary unit, √(−1)
√	square root of	the radical sign
π	pi	3.14159…
τ	tau	proper time
ω	omega	angular frequency
μ, ν	mu, new	spacetime indices, running 0–3
γ	gamma	the photon (written Qγ); also the Dirac matrices γ^μ
Δ	delta	the change in a quantity (Δe_0 = change in e_0)
Ĥ	H-hat	the energy operator (Hamiltonian)
ℂ	C	the complex numbers
ℍ	H	the quaternions (H for Hamilton)
⊗	tensor	combines two algebras: ℂ⊗ℍ is the bi-quaternions
$e_0 \ldots e_3$	e-naught … e-three	the four bi-quaternion axes
μ_B	mu-sub-B	the Bohr magneton (a unit of magnetic moment)
ζ_nℓ	zeta	radial spin-orbit coupling strength
j, m_j	jay, em-sub-jay	total angular momentum and its projection
g_J	gee-sub-jay	the Landé g-factor
S, L·S	ess, L-dot-S	spin vector; L·S is the spin-orbit coupling
σ	sigma	the 2×2 Pauli spin matrices; the model's $e_k = -i\,\sigma_k$