Spacetime Expressed Through a complex Quaternion

Special Relativity, General Relativity, and the Geometry of the Cosmos from Hamilton’s Algebra

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Abstract

We show that writing the time coordinate of spacetime as $W = i\tau$ — purely imaginary in the complex sense — on the real axis of Hamilton’s Quaternion, and placing the three spatial coordinates on the three Quaternion-imaginary axes ι, j, k, produces a complex Quaternion whose norm is the Lorentzian spacetime interval. This single algebraic choice is the one postulate of the framework: it replaces the usual signature convention and the separate invariance postulates. The complex unit i and the Quaternion units ι, j, k are independent: their squares both equal −1 but they are not the same −1. The Lorentz transformation emerges as the unique rotation of the complex Quaternion that preserves W as purely imaginary. Time dilation, length contraction, and $E = mc^2$ follow from the complex Quaternion norm and its Newtonian limit. Introducing mass through a scalar deformation f(r) of the metric and requiring zero curvature outside the mass uniquely determines $f(r) = 1 - r_s/r$, recovering the solution found by Schwarzschild in 1916. The predicted light deflection of 1.75 arcseconds in the 1919 eclipse and the Mercury perihelion precession of 43 arcseconds per century follow directly. We then show that empty space is not flat: the universe carries a background curvature with radius $R = c/H$. The full vacuum complex Quaternion metric is $f(r) = 1 - r_s/r - r^2/R^2$, containing both a local horizon at $r = r_s$ and a cosmological horizon at $r \approx R$. Both horizons are zeros of the same time component W of the same complex Quaternion. The cosmological term gives a redshift formula $1 + z = e^{d/R}$ developed in Paper 2 of this series.

Keywords: complex Quaternion, spacetime geometry, special relativity, general relativity, Lorentz transformation, vacuum metric, light deflection, perihelion precession, cosmological curvature, Hamilton algebra, Poincaré, Minkowski, Einstein

Chapter 1. Hamilton's Quaternion

1.1 Numbers that rotate

Most people know complex numbers, even if the name is unfamiliar. A complex number is simply a number with a real part and an imaginary part: a + bi, where $i^2 = -1$. It lives on a flat plane. The real part tells you where you are left-to-right; the imaginary part tells you where you are up-down. Geometrically, multiplying by i rotates you ninety degrees on that plane.

In 1843, William Rowan Hamilton spent years trying to extend complex numbers from two dimensions to three. He kept failing. The breakthrough came while walking with his wife along the Royal Canal in Dublin: you cannot do it in three dimensions. You need four. He carved the key equations into Brougham Bridge on the spot [10].

$$i^2 = j^2 = k^2 = ijk = -1$$

1.2 The four axes of the Quaternion

A Quaternion has one real part and three imaginary parts:

$$Q = W + X\iota + Yj + Zk$$

W, X, Y, Z are ordinary real numbers. The symbols ι (iota, the Greek letter corresponding to our i), j, and k are three independent imaginary units. We use ι to distinguish the first Quaternion imaginary from the complex unit i, which we will need separately. Hamilton's rules are:

$$\iota^2 = j^2 = k^2 = \iota jk = -1$$

Hamilton's multiplication table
$\iota j = k$ $j\iota = -k$ (order matters — Quaternions do not commute)
$jk = \iota$ $kj = -\iota$
$k\iota = j$ $\iota k = -j$
The three imaginary units cycle: $\iota \to j \to k \to \iota$
Non-commutativity is not a defect. It is why Quaternions describe 3D rotations exactly.

1.3 The norm

$$|Q|^2 = W^2 + X^2 + Y^2 + Z^2 \quad \text{(always non-negative for real W, X, Y, Z)}$$

The four-dimensional Pythagorean theorem. All terms positive.

1.4 The real axis is algebraically special

The three imaginary axes ι, j, k can be rotated freely into each other by transformations that preserve all the multiplication rules. These transformations are called automorphisms (Greek: autos = self, morphe = form). The real axis W cannot be rotated into any imaginary axis by any automorphism. The real part of a Quaternion is invariant under all automorphisms — it is structurally distinct from the three imaginary parts.

This algebraic fact will become, in the next chapter, the reason time is different from space.

Chapter 2. The complex Quaternion

2.1 The four coordinates of an event

An event in spacetime has four coordinates: when it happened (time t) and where it happened (three spatial coordinates x, y, z). We want to encode these four numbers into a single algebraic object. The quaternion has exactly four components. We write the spacetime displacement between two nearby events as:

$$dQ = W + dx \cdot \imath + dy \cdot j + dz \cdot k$$

Space goes on the three imaginary axes ı, j, k. Time goes on the real axis W. But what is W, exactly? This is the central question of the paper.

2.2 Why time has a minus sign — a heuristic preview

In Euclidean geometry, the distance between two nearby points is always positive:

$$ds^2 = dx^2 + dy^2 + dz^2 \quad (\text{always} \geq 0)$$

In spacetime, the interval is different. Famously, the time coordinate enters with a minus sign:

$$ds^2 = -c^2\, dt^2 + dx^2 + dy^2 + dz^2$$

This is the Lorentz Interval. Named after Hendrik Antoon Lorentz for historical reasons[i] — the interval is invariant under his transformations named Lorentz transformations which was a geometric formulation of spacetime as a four-dimensional manifold by Minkowski, 1908.

Standard treatments simply declare this sign as a definition, or say "the metric has signature (−,+,+,+)". The quaternion framework gives a reason. The Lorentzian interval is the quaternion norm with a sign flip on the real part:

$$ds^2 = -W^2 + X^2 + Y^2 + Z^2$$

The real axis and the imaginary axes are algebraically orthogonal in quaternion algebra — they cannot be mixed by any internal rotation of the quaternion. The minus sign on time is not arbitrary: time sits on the real axis and space sits on the three imaginary axes. At this stage, however, this is a heuristic picture — it locates where the minus sign will live, but it does not yet derive it. Sections 2.4–2.6 make it precise: writing W = ic·dτ turns the sign flip into an algebraic identity.

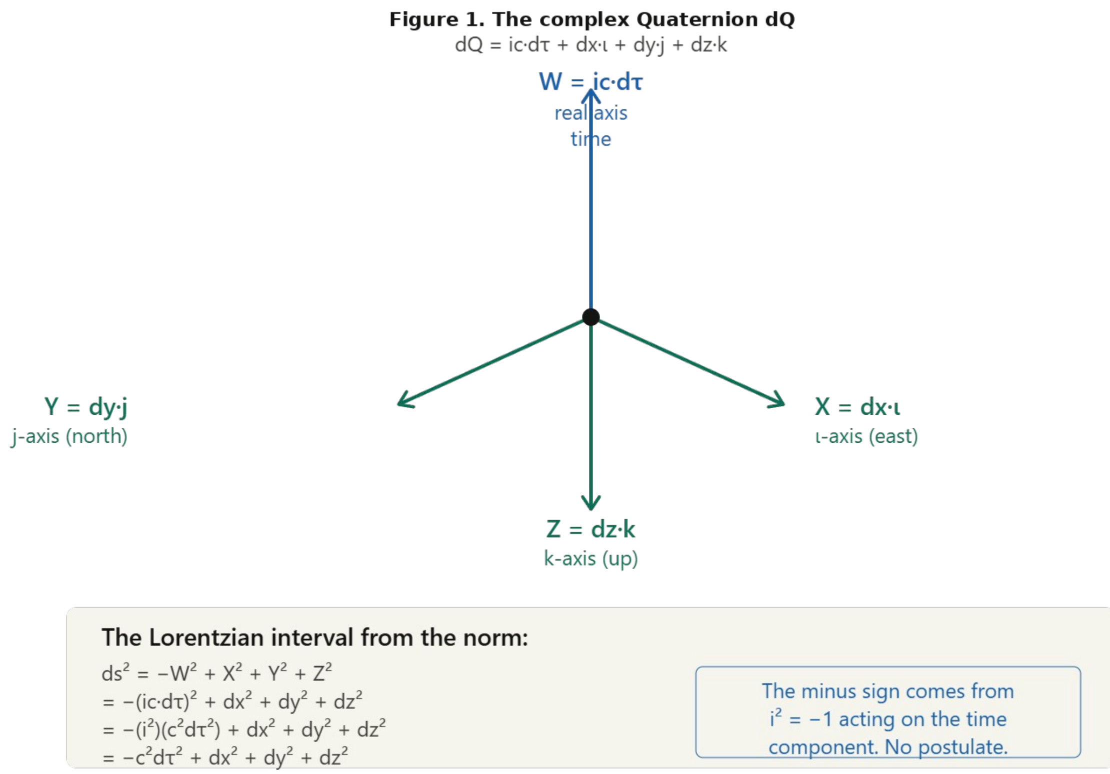

Figure 1. The quaternion dQ shown as a four-component object. The real axis W carries time. The three imaginary axes carry the three spatial directions. The Lorentzian interval $ds^2 = -W^2 + X^2 + Y^2 + Z^2$ falls directly out of this structure. The minus sign is derived algebraically in Section 2.6 from the single input W = ic·dτ.

2.3 Two check cases

A photon travels at speed c. In one second, it covers c metres. Take one second as dt and c metres as dx, with dy = dz = 0:

$$ds^2 = -(c \cdot dt)^2 + (c \cdot dt)^2 = 0$$

The real and imaginary parts cancel exactly. A photon has zero spacetime interval. It exists outside of time, which is why photons do not age. This follows automatically from the quaternion structure: the real axis carries the same magnitude as one imaginary axis, so they cancel under the Lorentzian norm.

An observer sitting still has dx = dy = dz = 0, so X = Y = Z = 0. The quaternion dQ is pure real. The interval is:

$$ds^2 = -(c \cdot dt)^2 < 0$$

Negative. This is the signature of a timelike interval: the journey is through time rather than space. The faster you move through space (larger X, Y, Z), the smaller the magnitude of ds^2 becomes, and the less time passes on your clock. This is relativistic time dilation, and it too follows from the quaternion structure.

2.4 Poincaré's insight: time is imaginary

In 1905, Henri Poincaré submitted a paper to the Rendiconti del Circolo Matematico di Palermo entitled Sur la dynamique de l'électron (On the dynamics of the electron) [1]. In it, he noticed something remarkable. If you write the time coordinate as:

$$l = ict \qquad \text{where } i = \sqrt{-1} \text{ (the complex unit) and } c = \text{speed of light}$$

then a Lorentz transformation (the rule for changing from one moving observer to another) becomes an ordinary rotation in a four-dimensional Euclidean space with coordinates (x, y, z, l). The invariant distance in that space is:

$$x^2 + y^2 + z^2 + l^2 = x^2 + y^2 + z^2 + (ict)^2 = x^2 + y^2 + z^2 - c^2t^2$$

The minus sign on time appears automatically, as $i^2 = -1$. Poincaré did not postulate the Lorentzian signature — he derived it from the algebraic nature of the time coordinate.

Hermann Minkowski developed this into a full four-dimensional geometry of spacetime in 1908 [2], and the ict notation appeared in textbooks for decades. It was eventually abandoned because it breaks when spacetime is curved (general relativity requires non-Euclidean geometry that the complex-time trick cannot handle). But Poincaré's core insight survived: the minus sign on time is algebraic, not postulated.

2.5 The complex Quaternion: two kinds of imaginary

The standard quaternion has a real number on the real axis: $W \in \mathbb{R}$. But there is nothing in Hamilton's multiplication rules that forbids W from being complex. If you allow $W \in \mathbb{C}$, you get what are called **complex Quaternions** — quaternions with complex coefficients:

$$Q = (a + bi) + (c + di)\cdot\imath + (e + fi)\cdot j + (g + hi)\cdot k$$

where a, b, c, d, e, f, g, h are all real numbers. Hamilton himself studied these mathematical structures in his later publications such as his book *Elements of Quaternions*. They are also called **biquaternions** or the **Clifford algebra**

Cl(3,0). Essentially, a complex quaternion combines real and imaginary scalars, which mathematically creates an eight-dimensional algebraic space.

We now combine Poincaré's insight with Hamilton's algebra. We set the real component of the quaternion to be purely imaginary in the complex sense:

```
W  =  iτ          where τ (tau, Greek letter) is a real number,
                          and i = √−1
```

The full spacetime displacement quaternion is then:

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dQ  =  ic·dτ  +  dx·ι  +  dy·j  +  dz·k
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We now have two different kinds of imaginary unit in one expression. We must be precise about what they are and how they differ.

Two kinds of imaginary — named and distinguished
Complex unit i: the imaginary unit of ordinary complex numbers, $i^2 = -1$.
It is a scalar. It commutes with everything.
It acts only on the time component $W = ic \cdot d\tau$.

Quaternion unit ι (iota): first imaginary axis of Hamilton's algebra, $\iota^2 = -1$.
It does not commute with j and k.
It acts on the spatial component dx.

Quaternion unit j (jay): second imaginary axis, $j^2 = -1$. Acts on dy.
Quaternion unit k (kay): third imaginary axis, $k^2 = -1$. Acts on dz.

The complex i and the quaternion ι, j, k are independent objects.
Their squares all equal −1, but they are not the same −1.
They inhabit different algebraic layers and do not interact.

2.6 The Lorentzian metric falls out

Compute the norm of dQ. Because the complex i commutes with the quaternion units, and because $W = ic \cdot d\tau$ while the spatial components are quaternion-imaginary, the norm gives:

$$
\begin{aligned}
|dQ|^2 &= W^2 + (dx)^2 + (dy)^2 + (dz)^2 \\
&= (ic \cdot d\tau)^2 + dx^2 + dy^2 + dz^2 \\
&= -c^2 d\tau^2 + dx^2 + dy^2 + dz^2
\end{aligned}
$$

This is the Lorentzian spacetime interval. The minus sign on time is $(ic \cdot d\tau)^2 = i^2c^2d\tau^2 = -c^2d\tau^2$. No separate signature postulate is needed: the sign follows from the single input $W = ic \cdot d\tau$.

The Lorentzian metric from the complex quaternion norm
$ds^2 = -c^2d\tau^2 + dx^2 + dy^2 + dz^2$

> This is the foundation of all relativistic physics.
> It follows from $W = ic \cdot d\tau$ and the properties of i.
> The choice $W = ic \cdot d\tau$ is the single postulate; nothing further is assumed.

A remark on the norm is needed here. For real components, $|Q|^2 = W^2 + X^2 + Y^2 + Z^2$ is always non-negative (Section 1.3). Once W is allowed to be complex, that guarantee no longer holds: $W^2 + X^2 + Y^2 + Z^2$ becomes the complexified norm form — a complex bilinear expression that is no longer positive-definite. The computation above evaluates it for W purely imaginary, and that is exactly the case that produces the Lorentzian signature, with negative values for timelike intervals.

A second remark concerns the word "postulate." The algebra above is exact, but the choice $W = i\tau$ rather than $W = \tau$ is itself the postulate of this framework — relocated from geometry into algebra, not eliminated. What is gained is economy and naturalness: one algebraic choice replaces the signature convention (−,+,+,+) and, as Chapter 3 shows, the separate invariance postulates. A related point: the transformations that preserve the complexified norm form a larger group than the Lorentz group (essentially the complex rotations); it is the additional requirement that W remain purely imaginary — stated explicitly in Section 3.2 — that cuts this larger symmetry down to exactly the Lorentz group. That requirement is part of the physical input.

2.7 Why the imaginary character of time is permanent

One might ask: could a physical process or mathematical transformation give time a real part, destroying the purely imaginary character of W? The answer is no, and the reason is a theorem, not an assumption.

The first wall: centrality

The complex unit i is a central element of the complex quaternion algebra. Central means: it commutes with every element of the algebra, including all quaternion units ı, j, k and all real numbers. No multiplication by a quaternion unit, no rotation in the ıjk space, can reach inside $W = ic \cdot d\tau$ and produce a real part. The quaternion operations see $ic \cdot d\tau$ as a single inert object and act only on $d\tau$, which is real, leaving W purely imaginary.

The second wall: the Lorentz group

The physical transformations that act on the time coordinate are Lorentz transformations: time dilation, boosts between observers moving at different speeds. These transformations act on τ by real rescaling: τ is replaced by γτ (gamma tau), where γ (gamma, Greek letter) is a real positive number called the Lorentz factor. A real number multiplied by a real factor stays real. So τ stays real under all Lorentz transformations, and therefore $ic \cdot d\tau$ stays purely imaginary.

Together these two walls mean: no operation internal to the algebra, and no physical transformation within special or general relativity, can give the time component W a real part. The imaginary character of time is structurally permanent.

2.8 The connection to Poincaré and Minkowski

Poincaré's notation $l = ict$ is now recognizable as the statement $W = ic \cdot d\tau$: the real component of the spacetime quaternion is the complex unit i times a real coordinate. His ict was not a mathematical trick; it was the correct identification of the time coordinate as a complex-imaginary scalar sitting on the real axis of what would have been, had he had the language, a complex quaternion.

Minkowski's four-dimensional spacetime is the geometry of this complex quaternion. His Lorentz transformations are the real automorphisms of the complex quaternion that preserve τ as real (and therefore W as purely imaginary). The reason ict failed in curved spacetime is that complex-number tricks do not generalize to non-Euclidean geometry, but the complex quaternion structure does: the metric function f(r) introduced in Chapter 4 scales $ic \cdot d\tau$ to $i\sqrt{f} \cdot c \cdot d\tau$, which is still purely imaginary, and the algebra holds throughout.

A note on priority: Quaternion formulations of special relativity are not themselves new. Arthur Conway (1911) [11] and Ludwik Silberstein (1912) [12] developed biquaternion treatments of the Lorentz transformation, and Silberstein's 1914 textbook presents the approach at length. The contribution of the present series is not the special-relativistic construction alone but the carrying of one and the same algebra through general relativity (Chapters 4–7), quantum mechanics (Papers 3 and 5), and cosmology (Paper 2).

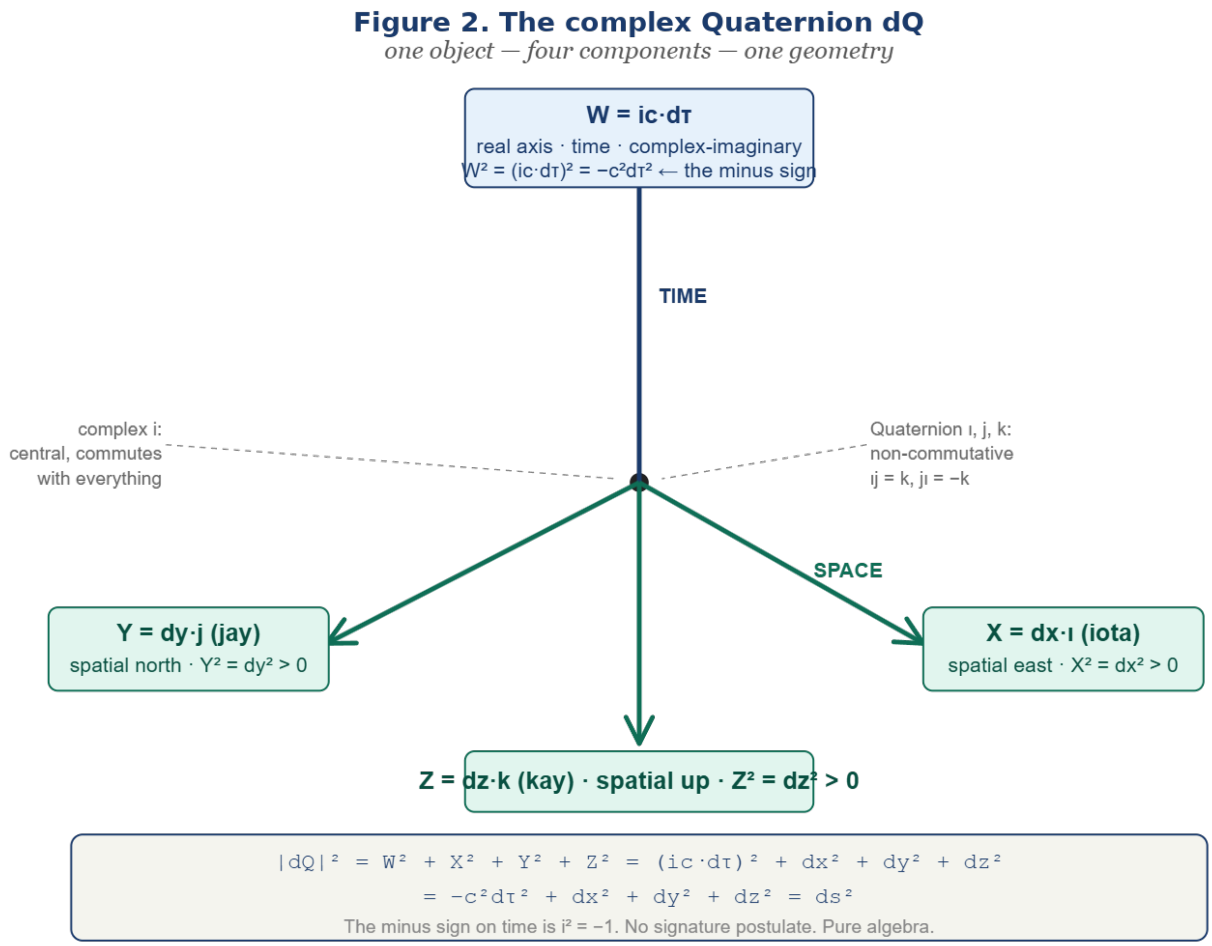


Figure 2. **The complex Quaternion $dQ = ic\cdot d\tau + dx\cdot ı + dy\cdot j + dz\cdot k$**
Time sits on the real axis as a complex-imaginary scalar $ic\cdot d\tau$.
Space sits on the three quaternion-imaginary axes ı, j, k.
The norm gives $ds^2 = -c^2d\tau^2 + dx^2 + dy^2 + dz^2$, the minus sign supplied by $(ic\cdot d\tau)^2 = -c^2d\tau^2$.
The two kinds of imaginary (complex i and quaternion ı, j, k) do not interact.

Chapter 3. Special Relativity from the complex Quaternion

Special relativity describes how two observers moving at constant velocity relative to each other measure the same physical events. Einstein's 1905 formulation [3] required two postulates: the laws of physics are the same for all inertial observers, and the speed of light is the same for all observers. We derive the consequences here directly from the complex quaternion: the single algebraic input $W = ic\cdot d\tau$, together with the requirement that rotations preserve W as purely imaginary, does the work of those two postulates.

3.1 What ds^2 is and why it is the same for all observers

We need to be precise about something that the paper has been using without fully explaining. What exactly is ds^2, and why should it be the same for two observers who are measuring different times and distances?

What ds^2 is

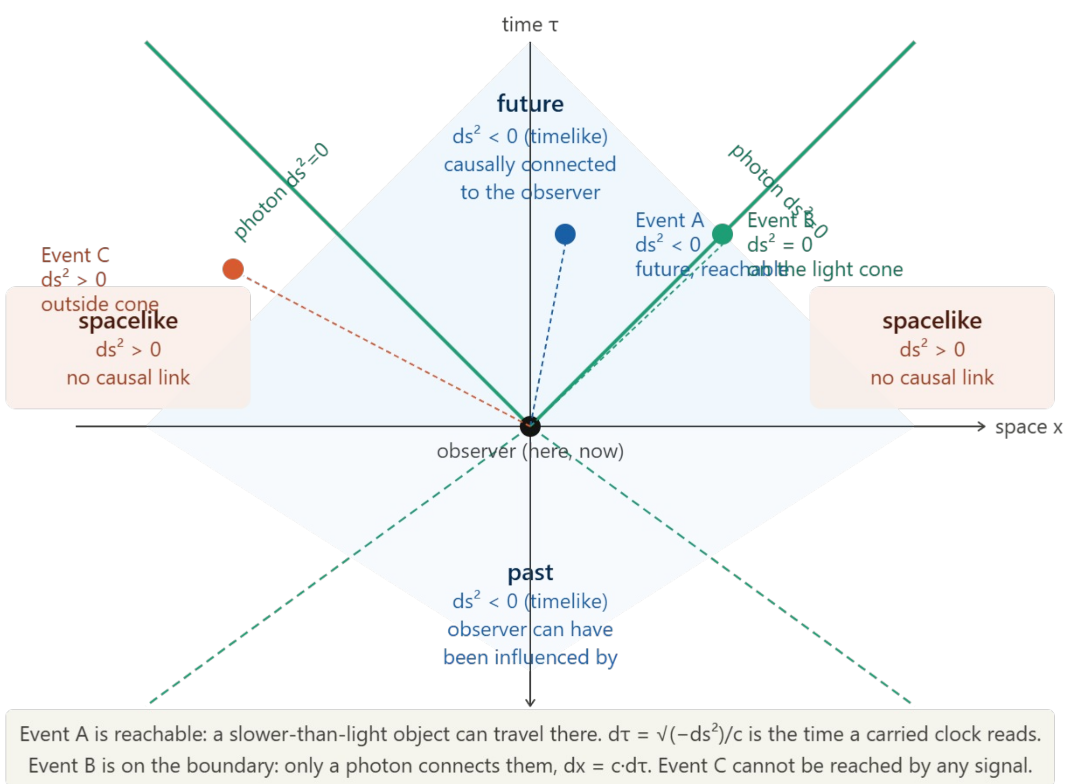
An event is something that happens at a definite place and a definite time: a firecracker explodes, a photon is emitted, a clock ticks. Two events have a spacetime separation — not just a spatial distance and not just a time difference, but a combined four-dimensional gap between them.

The spacetime interval ds^2 is that combined gap, computed as:

$$ds^2 = -c^2 d\tau^2 + dx^2 + dy^2 + dz^2$$

The time part enters with a minus sign (from the complex quaternion: $W^2 = (icd\tau)^2 = -c^2 d\tau^2$). The spatial parts enter with positive signs. The result ds^2 can be positive, negative, or zero, and each case has a physical meaning:

Figure 3. The light cone
the spacetime diagram that makes the three intervals visible



> **$ds^2 < 0$ (timelike):** Time dominates. The two events are causally connected. A physical object or signal moving slower than light can travel between them.

A clock carried from one event to the other would tick a proper time $d\tau = \sqrt{(-ds^2)}/c$. All observers agree on which event came first.

$ds^2 = 0$ (lightlike, or null): The two events are exactly on each other's light cone. Only a photon traveling at c connects them. The spatial distance equals c times the time gap: $dx = c \cdot d\tau$. This is the boundary between timelike and spacelike.

$ds^2 > 0$ (spacelike): Space dominates. No physical signal can connect the two events. Different observers may disagree on which came first. There is no causal relationship between them.

A concrete example

Event 1: a light bulb switches on at position x = 0, time τ = 0.

Event 2: the same light pulse arrives at position x = c metres, time τ = 1 second.

Observer A (stationary): measures dτ = 1 second, dx = c metres.

$$ds^2 = -c^2(1)^2 + (c)^2 = -c^2 + c^2 = 0$$

Observer B (moving at speed v = c/2 relative to A): time dilation and length contraction change their measurements. They measure dτ' ≠ 1 second and dx' ≠ c metres. But when they compute:

$$ds'^2 = -c^2 d\tau'^2 + dx'^2 = 0$$

Also zero. Both observers agree: these two events are connected by a light ray. That physical fact — light travels at c for everyone — is encoded in the invariance of ds^2.

Why the complex quaternion norm is invariant — the algebraic reason

The complex quaternion of Observer A is dQ_A. The complex quaternion of Observer B is dQ_B. The two are related by a Lorentz transformation, which is a rotation of the complex quaternion — a hyperbolic rotation in the W-ι plane (the time-space plane) that preserves W = ic·dτ as purely imaginary.

Here is the key fact from quaternion algebra: the norm of a quaternion does not change under rotation. A rotation rearranges the components of a quaternion but preserves their combined squared magnitude. This is exactly analogous to ordinary geometry: rotating a ruler does not change its length.

Therefore:

$$|dQ_A|^2 = |dQ_B|^2$$

$$W_A^2 + X_A^2 + Y_A^2 + Z_A^2 = W_B^2 + X_B^2 + Y_B^2 + Z_B^2$$

$$-c^2 d\tau\_A^2 + dx\_A^2 + dy\_A^2 + dz\_A^2 \quad = \quad -c^2 d\tau\_B^2 + dx\_B^2 + dy\_B^2 + dz\_B^2$$

$$ds^2\_A \quad = \quad ds^2\_B$$

The individual components dτ, dx, dy, dz change from A to B. But the combination ds^2 does not. It is the quaternion norm, and the quaternion norm is preserved by rotation.

This is why special relativity works. Every observer has their own complex quaternion, related to every other observer's by a quaternion rotation. The norm is invariant under rotation. Therefore every observer measures the same ds^2 for any pair of events. The spacetime interval is an objective, observer-independent quantity. Everything else — times, distances, velocities — depends on who is measuring. The interval does not.

> **The invariant interval — why it matters**
> $ds^2 = -c^2d\tau^2 + dx^2 + dy^2 + dz^2$ is the same for all observers.
> Reason: it is the complex quaternion norm, and norms are preserved under rotation.
> A Lorentz transformation is a rotation. Therefore it preserves ds^2.
>
> Individual measurements that differ between observers:
> dτ (time between events) — depends on observer's speed
> dx (spatial gap between events) — depends on observer's speed
>
> What all observers agree on:
> ds^2 (the combined spacetime interval)
> The sign of ds^2 (timelike / null / spacelike)
> Whether two events are causally connected

3.2 The Lorentz transformation as a quaternion rotation

Consider two observers: one at rest (we call them Observer A) and one moving at speed v along the x-axis (Observer B). When they observe the same event, their coordinates are related by a Lorentz boost. We derive this from the complex quaternion.

A boost along the x-axis mixes the time component W and the ι-component (the x-direction). In the complex quaternion, this is a rotation in the W–ι plane. The rotation must preserve the imaginary character of W = ic·dτ — that is the constraint that makes it a Lorentz transformation rather than an ordinary Euclidean rotation.

An ordinary rotation in a plane is written using a trigonometric angle θ (theta, Greek letter). But the W–ι plane is mixed: W is complex-imaginary, ι is quaternion-imaginary. The rotation that preserves the Lorentzian norm uses hyperbolic functions instead of trigonometric ones. Define the rapidity φ (phi, Greek letter) by:

$$\tanh(\varphi) = v/c \qquad \text{where tanh is the hyperbolic tangent function}$$

Then the Lorentz boost is:

$$c\cdot d\tau' = \cosh(\varphi)\cdot c\cdot d\tau - \sinh(\varphi)\cdot dx$$

$$dx' = -\sinh(\varphi)\cdot c\cdot d\tau + \cosh(\varphi)\cdot dx$$

$$dy' = dy \qquad dz' = dz$$

where cosh and sinh (pronounced "co-shine" and "shine") are the hyperbolic cosine and sine. The primed coordinates belong to Observer B. Verify that this preserves ds^2:

$$-c^2 d\tau'^2 + dx'^2 = -(\cosh\varphi\cdot cd\tau - \sinh\varphi\cdot dx)^2 + (-\sinh\varphi\cdot cd\tau + \cosh\varphi\cdot dx)^2$$

$$= (\cosh^2\varphi - \sinh^2\varphi)(-c^2 d\tau^2 + dx^2)$$

$$= -c^2 d\tau^2 + dx^2 \qquad \text{since } \cosh^2\varphi - \sinh^2\varphi = 1$$

The invariant is preserved. This is the Lorentz transformation, derived from the requirement that the complex quaternion rotation preserve both the norm and the imaginary character of the time component.

Converting rapidity to velocity using $\tanh(\varphi) = v/c$ and the identities $\cosh(\varphi) = \gamma$, $\sinh(\varphi) = \gamma v/c$ where:

$$\gamma = 1 / \sqrt{(1 - v^2/c^2)} \qquad \text{(Lorentz factor, always} \geq 1\text{)}$$

gives the familiar form:

$$d\tau' = \gamma(d\tau - v\cdot dx/c^2)$$

$$dx' = \gamma(dx - v\cdot d\tau)$$

3.3 Time dilation

Consider Observer B carrying a clock that ticks at their own location: $dx = 0$ in their frame (B is always at the same place as their clock). Substituting $dx = 0$ into the invariant interval from Observer A's perspective:

$$ds^2 = -c^2 d\tau_A^2 + v^2 d\tau_A^2 = -c^2 d\tau_B^2$$

(using $dx = v\cdot d\tau_A$ since B moves at speed v relative to A). Solving:

$$d\tau_B = d\tau_A \times \sqrt{(1 - v^2/c^2)} = d\tau_A / \gamma$$

Observer B's clock ticks more slowly than Observer A's by the Lorentz factor γ. This is time dilation. It follows directly from the complex quaternion norm: the purely imaginary time component $W = ic \cdot d\tau$ has its magnitude reduced when some of the interval is "used up" by spatial motion.

3.4 Length contraction

Consider a rod of rest length L_0 lying along the x-axis in Observer B's frame. Observer A measures it by noting the positions of both ends simultaneously ($d\tau_A = 0$). Setting $d\tau_A = 0$ in the invariant:

$$ds^2 = dx_A^2 = -c^2 d\tau_B^2 + L_0^2$$

Solving for the length measured by A:

$$L_A = L_0 / \gamma = L_0 \times \sqrt{(1 - v^2/c^2)}$$

The rod appears shorter to A by the Lorentz factor. Length contraction is the spatial counterpart of time dilation. Both follow from the same complex quaternion norm.

3.5 The energy-momentum complex quaternion and $E = mc^2$

Every object in spacetime has two complex quaternions associated with it. The first is the spacetime displacement dQ, which we have been studying. The second is the energy-momentum complex quaternion p, which encodes how much energy and momentum the object carries. These two complex quaternions are partners: one describes where and when, the other describes how much energy and motion.

Building the energy-momentum complex quaternion

We write p with energy on the real axis and the three momentum components on the quaternion-imaginary axes:

$$p = (iE/c) + p_x \cdot \imath + p_y \cdot j + p_z \cdot k$$

Why does energy sit on the real axis with a complex i in front of it? For the same reason that time does. Energy and time are conjugate variables — their product has units of action (joules times seconds, the unit of Planck's constant ħ). Conjugate variables share their algebraic character. Time is $ic \cdot d\tau$ (purely imaginary complex); energy must be iE/c (purely imaginary complex). The two walls that protect the imaginary character of time — centrality of i and the Lorentz group acting by real rescaling — protect the energy component in exactly the same way.

The three momentum components p_x, p_y, p_z (momentum in the x, y, z directions) sit on the three quaternion-imaginary axes ı, j, k, exactly as the spatial displacement components dx, dy, dz do.

The norm of p is also invariant

By the same argument as for ds^2: a Lorentz transformation is a quaternion rotation. The norm of a quaternion is preserved under rotation. Therefore the norm of p is the same for all observers. Let us compute it:

$$|p|^2 = (iE/c)^2 + p_x^2 + p_y^2 + p_z^2$$

$$= -E^2/c^2 + |\rho|^2$$

where $|\rho|^2 = p_x^2 + p_y^2 + p_z^2$ is the squared magnitude of the three-momentum. The minus sign on E^2 comes from $(iE/c)^2 = i^2E^2/c^2 = -E^2/c^2$, exactly as the minus sign on time came from $(ic\cdot d\tau)^2 = -c^2 d\tau^2$.

Evaluating the norm in the rest frame

The norm $|p|^2$ is invariant — every observer computes the same value. So we are free to choose the simplest observer to compute it: the one who sees the particle sitting still.

An observer moving with the particle (the rest frame) sees zero momentum: $p_x = p_y = p_z = 0$, so $|\rho| = 0$. The particle has some energy — call it E_0 (E-naught), the rest energy. The norm in the rest frame is:

$$|p|^2\_rest = -E_0^2/c^2 + 0 = -E_0^2/c^2$$

Since the norm is invariant, this equals the norm in any other frame:

$$-E^2/c^2 + |\rho|^2 = -E_0^2/c^2$$

$$E^2 = E_0^2 + |\rho|^2 c^2$$

This is the full relativistic energy-momentum relation. We still need to know what E_0 is. That requires one more argument.

What is the rest energy? The Newtonian limit

At low velocities ($v \ll c$, meaning v much smaller than c), special relativity must reduce to Newtonian mechanics. In Newtonian mechanics, the kinetic energy of a moving particle is $\frac{1}{2}mv^2$ and the momentum is mv.

From the energy-momentum relation, the total energy of a slowly moving particle ($|\rho| = mv$ at low v) is:

$$E = \sqrt{(E_0^2 + m^2v^2c^2)}$$

$$\approx E_0 \times \sqrt{(1 + m^2v^2c^2/E_0^2)}$$

$$\approx E_0 + m^2v^2c^2/(2E_0) \qquad \text{(using } \sqrt{(1+x)} \approx 1 + x/2 \text{ for small x)}$$

For this to match the Newtonian kinetic energy $\frac{1}{2}mv^2$, we need the second term to equal $\frac{1}{2}mv^2$:

$$m^2v^2c^2 / (2E_0) = \tfrac{1}{2}mv^2$$

$$E_0 = mc^2$$

The rest energy of a particle is its mass times c squared. This is not a postulate. It is the unique value of E_0 that makes the relativistic energy-momentum relation consistent with Newtonian mechanics at low velocity.

$E = mc^2$ and the full relation

Substituting $E_0 = mc^2$ into the norm:

$$|p|^2 = -E_0^2/c^2 = -(mc^2)^2/c^2 = -m^2c^2$$

This is where $-m^2c^2$ comes from. It is not assumed; it is derived from the Newtonian limit. The norm of the energy-momentum complex quaternion is $-m^2c^2$ because the rest energy is mc^2, which is forced by consistency with low-velocity physics.

The full energy-momentum relation is:

$$E^2 = m^2c^4 + |\rho|^2c^2$$

For a particle at rest ($|\rho| = 0$): $E = mc^2$. For a particle moving at speed v: $E = \gamma mc^2$ where $\gamma = 1/\sqrt{(1-v^2/c^2)}$ is the Lorentz factor. For a massless photon ($m = 0$): $E = |\rho|c$.

$E = mc^2$ from the complex quaternion — the complete chain

1. Energy sits on the real axis as iE/c (conjugate to time, same algebraic character).
2. The norm of p is invariant under Lorentz transformation (same as ds^2).
3. In the rest frame: norm = $-E_0^2/c^2$.
4. Setting norm = $-E_0^2/c^2$ and requiring Newtonian mechanics at low v:
   $m^2v^2c^2/(2E_0) = \frac{1}{2}mv^2 \rightarrow E_0 = mc^2$.
5. Therefore norm = $-m^2c^2$, and $E^2 = m^2c^4 + |\rho|^2c^2$.
6. At rest: $E = mc^2$.

No step is assumed. Each follows from the previous.

3.6 The photon

A photon has zero rest mass: $m = 0$. Its energy-momentum complex quaternion norm is:

$$-E^2/c^2 + |\rho|^2 = 0 \quad \Rightarrow \quad E = |\rho|\cdot c$$

The energy equals the momentum magnitude times c. This is the photon dispersion relation. In the spacetime picture, a photon travels along the light cone $ds^2 = 0$, meaning its complex quaternion interval has zero norm. The complex-imaginary time component and the quaternion-imaginary spatial component exactly cancel. A photon exists at the boundary between the timelike and spacelike worlds — the zero-norm surface of the complex quaternion.

Special relativity from the complex quaternion — summary
Invariant interval: $ds^2 = -c^2d\tau^2 + dx^2 + dy^2 + dz^2$ (from the norm)
Lorentz transformation: hyperbolic rotation preserving $W = ic\cdot d\tau$ as imaginary
Time dilation: $d\tau\_B = d\tau\_A / \gamma$ (from ds^2 invariance, $dx = 0$)
Length contraction: $L\_A = L_0 / \gamma$ (from ds^2 invariance, $d\tau\_A = 0$)
$E = mc^2$: from the norm of the energy-momentum complex quaternion
Photon: $ds^2 = 0$, zero-norm complex quaternion, $E = |\rho|c$
All of these follow from the single object dQ and its norm.
No separate postulates about light speed or observer equivalence are needed.

Chapter 4. Curved Space and the Geometry of Mass

Mass curves spacetime. In the complex Quaternion framework this statement has a precise and simple meaning: mass changes the function f(r) that deforms the time and radial components of the complex Quaternion. Everything about gravity — orbits, light bending, time dilation, black holes — is encoded in one scalar function of one variable.

4.1 The geometry in one function

In flat empty space the complex Quaternion is:

$$dQ = ic\cdot d\tau + dx\cdot \imath + dy\cdot j + dz\cdot k$$

$f = 1$ everywhere. Introduce a spherical mass M. By spherical symmetry, mass can only affect the radial direction. The deformed complex Quaternion is:

$$dQ = i\sqrt{f}\cdot c\cdot d\tau + (1/\sqrt{f})\cdot dr\cdot \imath + r\cdot d\theta\cdot j + r\cdot \sin\theta\cdot d\varphi\cdot k$$

The entire effect of mass on spacetime geometry is contained in one scalar function f(r). The angular components $r\cdot d\theta\cdot j$ and $r\cdot\sin\theta\cdot d\varphi\cdot k$ are untouched — mass does not affect sideways distances. Only the time component and the radial component are deformed, and they are deformed reciprocally: $\sqrt{f}$ on time, $1/\sqrt{f}$ on radius. Their product is always 1. We take this exact reciprocity as an ansatz — the deformation that preserves the algebraic structure — and the vacuum equations of Section 4.3 confirm it: the reciprocity $g\_tt\cdot g\_rr = -c^2$ is a consequence of the field equations in this setting, not something the algebra alone forces.

When $f < 1$: clocks run slow ($\sqrt{f} < 1$) and radial rulers stretch ($1/\sqrt{f} > 1$). This is gravitational time dilation and radial length contraction — the two effects that produce light bending, perihelion precession, and gravitational redshift. All from one number f(r) at each point.

4.2 What curvature means for f(r)

The curvature of the complex Quaternion is the degree to which f(r) deviates from 1 and changes with position. If $f = 1$ everywhere, the geometry is flat — Minkowski space, no gravity. If f varies with r, the geometry is curved — gravity is present.

The geometric statement that the complex Quaternion curvature vanishes in empty space outside a mass is a condition on f(r) and its derivatives. The machinery of Christoffel symbols and Riemann tensors is how that condition is extracted from the metric by standard tensor calculus, and that computation must be carried out once for this ansatz. What the complex Quaternion contributes is the encoding: in it, f(r) already is the complete curvature information, packaged in a single scalar function.

The condition is simply this: outside the mass, where there is no matter and no energy, the geometry must be as smooth as possible. The mathematical statement of maximum smoothness for a spherically symmetric f(r) is:

```
d/dr [ r · f(r) ]  =  1     … (the vacuum condition)
```

Read it aloud: the derivative of r times f with respect to r equals 1. One equation. One derivative. This is all that Einstein's 16 tensor field equations reduce to when written for a spherically symmetric vacuum metric in complex Quaternion language.

The entire apparatus of Christoffel symbols, Riemann curvature tensor, Ricci tensor, and Einstein tensor is the mathematical route Einstein had to take because he was working with a 4×4 tensor metric rather than a single scalar function. In the complex Quaternion, f(r) already is the geometry: once the standard curvature computation has been performed for this ansatz, the intermediate steps collapse into the single equation above.

4.3 Solving the vacuum condition

Integrating d/dr[r·f] = 1:

```
r · f(r)  =  r  −  C

f(r)  =  1  −  C/r
```

where C is a constant of integration. Two conditions fix C.

First: space must be flat far from the mass. As $r \to \infty$, $f \to 1$. This is already satisfied for any C.

Second: a slowly moving planet must feel Newton's gravitational acceleration $-GM/r^2$ at large distances. Working out the geodesic — the straightest path in the curved complex Quaternion geometry — and taking the low-velocity limit gives an acceleration of $-Cc^2/(2r^2)$. Setting this equal to $-GM/r^2$:

```
C  =  2GM/c²  ≡  r_s     (the Schwarzschild radius)
```

Therefore:

$$f(r) = 1 - r_s/r \qquad \text{where} \quad r_s = 2GM/c^2$$

This is the Schwarzschild solution [4]. Schwarzschild found it in 1916 by solving 16 coupled tensor equations from the Russian front. We find it by integrating one equation and applying two boundary conditions. Two paths. The same result.

4.4 The complete vacuum complex Quaternion metric

Substituting f(r) = 1 − r_s/r:

$$dQ = i\sqrt{(1-r_s/r)}\cdot c\cdot d\tau + dr/\sqrt{(1-r_s/r)}\cdot \iota + r\cdot d\theta\cdot j + r\cdot \sin\theta\cdot d\varphi\cdot k$$

The spacetime interval is:

$$ds^2 = -(1-r_s/r)c^2 d\tau^2 + dr^2/(1-r_s/r) + r^2 d\theta^2 + r^2 \sin^2\theta\cdot d\varphi^2$$

Two limits are immediate without any further calculation.

As $r \to \infty$: $f \to 1$. The complex Quaternion becomes flat Minkowski space. Gravity disappears at large distances.

As $r \to r_s$: $f \to 0$. The time component $W = i\sqrt{f}\cdot c\cdot d\tau \to 0$. The complex Quaternion loses its time component entirely. No time passes at this surface as seen from outside. This is the event horizon — the zero of the time component of the complex Quaternion — and it requires no separate argument to derive. It is already visible in f(r).

> **At the event horizon $r = r_s$**
> $f = 0 \;\Rightarrow\; \sqrt{f} = 0 \;\Rightarrow\; W = i\sqrt{f}\cdot c\cdot d\tau = 0$
> The complex Quaternion is purely Quaternion-imaginary. No time component.
> A clock at the horizon stops as seen from outside.
> The horizon is the zero of W — the same algebraic event as the quantum transition.
> In the quantum leap (Paper 5 [9]), the real part of the state Quaternion also passes through zero at the moment of transition. One algebra. Two scales.

4.5 Einstein’s field equation in complex Quaternion language

The vacuum condition $d/dr[r\cdot f] = 1$ is valid outside the mass in flat empty space. Inside matter, or with a cosmological background, the right side is modified.

With matter of energy density ρ(r):

$$d/dr[r\cdot f(r)] = 1 - (8\pi G/c^4)\cdot r^2\cdot \rho(r)$$

With cosmological background curvature R = c/H:

$$d/dr[r \cdot f(r)] = 1 - 3r^2/R^2$$

With both:

$$d/dr[r \cdot f(r)] = 1 - 3r^2/R^2 - (8\pi G/c^4) \cdot r^2 \cdot \rho(r)$$

This single ordinary differential equation in one scalar function f(r) is Einstein's field equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = (8\pi G/c^4) \cdot T_{\mu\nu}$ translated into complex Quaternion language. The cosmological constant $\Lambda = 3/R^2$ appears as the background curvature of empty space. The energy-momentum tensor $T_{\mu\nu}$ reduces, for a spherically symmetric distribution, to the single function ρ(r). The 16-component tensor equation becomes one scalar equation.

Integrating the full equation with the boundary conditions $f \to 1 - r^2/R^2$ as $r \to \infty$ and Newton's law at large r gives the complete vacuum metric:

$$f(r) = 1 - r_s/r - r^2/R^2$$

Both terms from one integration. Local gravity and cosmological curvature unified in one scalar function. The complex Quaternion contains both, and the single differential equation produces both simultaneously.

The complete geometry of mass in the complex Quaternion
One function f(r) encodes all of gravity.
One equation $d/dr[r \cdot f] = 1$ encodes the vacuum condition.
One integration gives $f(r) = 1 - r_s/r - r^2/R^2$.
Two horizons: $f = 0$ at $r = r_s$ (black hole) and $r \approx R$ (cosmos).
Both are zeros of the time component $W = i\sqrt{f} \cdot c \cdot d\tau$.
Same algebra. Two scales. One equation.

Chapter 5. Classical Tests

The vacuum complex Quaternion metric makes precise quantitative predictions that differ from Newton's gravity. Three predictions are the classical tests of general relativity. We work through two here. Gravitational redshift is the subject of Paper 2 [6].

5.1 Light deflection by the Sun — the 1919 eclipse

A photon has zero rest mass and travels on the zero-norm surface of the complex Quaternion: $ds^2 = 0$. In the vacuum metric, the zero-norm condition plus conservation of angular momentum gives a total deflection angle for a ray grazing the Sun:

$$\delta = 4GM_\odot / (c^2 \times R_\odot) = 2r_s / R_\odot$$

Why this is twice Newton's prediction

Newton treated light as a particle in the gravitational potential $-GM/r$. His prediction is $\delta_{Newton} = 2GM/c^2R = r_s/R$. The vacuum complex Quaternion metric gives exactly twice this. The factor of 2 has a precise origin: the metric has two components that bend the photon path. The time component $i\sqrt{f}\cdot c\cdot d\tau$, scaled by $\sqrt{f}$, contributes deflection r_s/R — this is the gravitational potential term Newton had. The radial component $dr/\sqrt{f}$, scaled by $1/\sqrt{f}$, contributes an equal additional r_s/R — this is the spatial metric distortion Newton had no concept of. Newton had a potential. He had no spatial metric. The complex Quaternion has both. They contribute equally and add.

Numbers

> **r_s (Sun):** $2GM_\odot/c^2 = 2 \times 6.674\times10^{-11} \times 1.989\times10^{30} / (3\times10^8)^2 = 2.95$ km
>
> **R☉:** 6.96×10^8 m
>
> **δ:** $2 \times 2950 / 696{,}000{,}000 = 8.48\times10^{-6}$ rad $= 8.48\times10^{-6} \times 206{,}265$ arcsec/rad $= 1.75$ arcseconds

> **1919 eclipse — Eddington at Príncipe, Crommelin at Sobral, 29 May 1919**
> complex Quaternion metric predicts: 1.75 arcseconds
> Newton predicts: 0.875 arcseconds (potential only, no spatial metric)
> Observed: 1.61 to 1.98 arcseconds
> The observation confirmed the complex Quaternion metric and ruled out Newton.
> Einstein became the world’s most famous scientist the next morning.

5.2 Mercury perihelion precession

Mercury’s orbit should be a closed ellipse under Newton’s pure $1/r^2$ force. Astronomers measured from 1859 that the perihelion (closest point to Sun) drifts forward by 574 arcseconds per century. Planetary perturbations account for 531 arcseconds. The remaining 43 arcseconds per century had no Newtonian explanation for sixty years. Einstein solved it on 18 November 1915, the week he completed general relativity. He said it gave him heart palpitations.

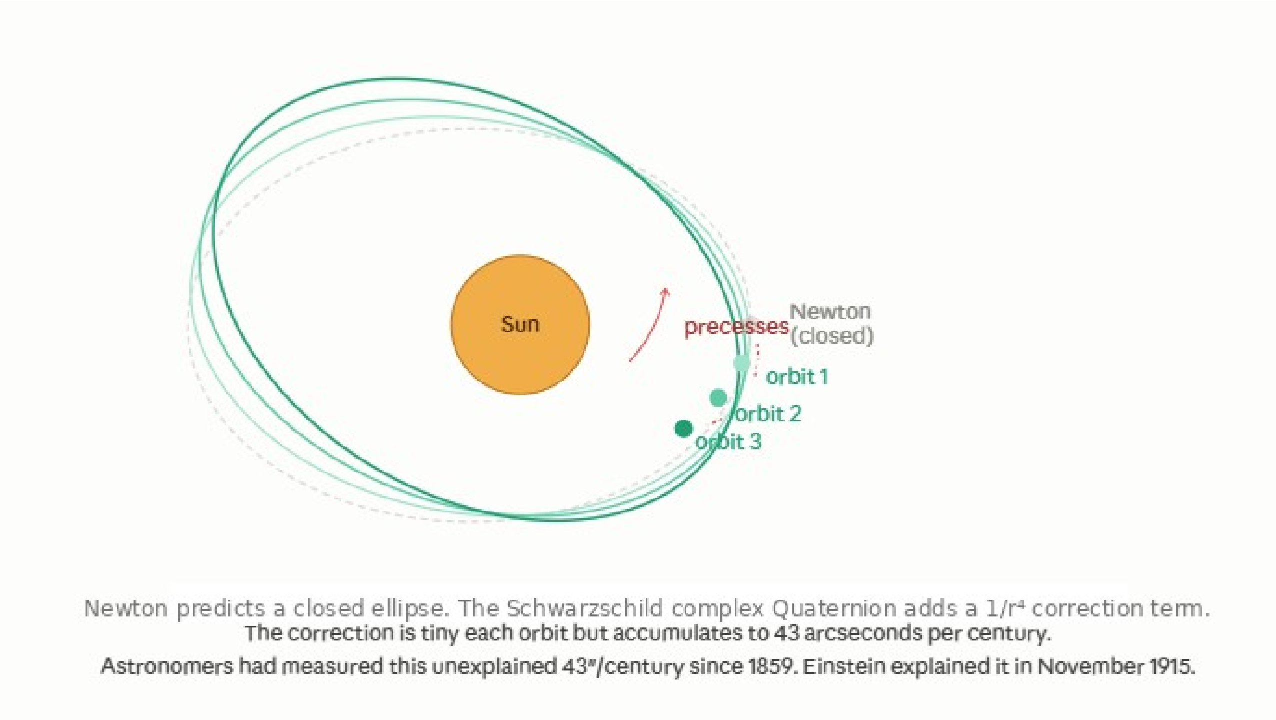


Figure 4. Mercury's perihelion precession. Newton predicts a closed ellipse; the $1/\sqrt{f}$ radial component of the complex Quaternion adds the small attractive $1/r^3$ potential term that makes the orbit precess by 43 arcseconds per century.

The origin in the complex Quaternion

The vacuum metric has a radial component $dr/\sqrt{f}$ rather than Newton's plain dr. This changes the effective potential seen by a planet. Expanding $f(r) = 1 - r_s/r$:

$$V_{eff} = -GM/r + L^2/(2r^2) - GML^2/(c^2r^3)$$

Here and throughout this section, L denotes angular momentum per unit mass (specific angular momentum) and the effective potential is likewise per unit mass; with this convention no factors of m appear.

Where each term comes from

A planet orbiting the Sun carries two conserved quantities throughout its orbit. The first is energy E — conserved because the Schwarzschild metric does not depend on time. The second is the angular momentum per unit mass L — conserved because the metric does not depend on the azimuthal angle φ. These two conservation laws, applied to the geodesic equation in the Schwarzschild metric, give a single equation for the radial motion:

$$(dr/d\tau)^2 = (E/mc^2)^2 - (1 - r_s/r)(1 + L^2/c^2r^2)$$

Expanding the right side and collecting terms by power of r, the effective potential emerges term by term:

First term: $-GM/r$

This is Newton's gravitational potential. It comes from the time component of the complex Quaternion: the factor $(1 - r_s/r) = (1 - 2GM/c^2r)$ in the metric. Newton had this. Every planet in every solar system calculation uses it.

Second term: $+L^2/(2r^2)$

This is the centrifugal barrier. It also exists in Newtonian mechanics — it is what prevents a planet with angular momentum from falling straight inward. It comes from the angular components of the metric: $r^2 d\varphi^2$ contributes L^2/r^2 to the energy budget. Newton had this too.

Third term: $-GML^2/(c^2r^3)$

This is the relativistic correction. Newton did not have it. It comes from the radial component of the complex Quaternion.

In Newton's theory the radial metric is plain dr. In the Schwarzschild complex Quaternion the radial component is $dr/\sqrt{f} = dr/\sqrt{(1 - r_s/r)}$. Expanding for small r_s/r:

$$dr/\sqrt{(1 - r_s/r)} \approx dr \cdot (1 + r_s/2r + \ldots)$$

The extra factor $r_s/2r$ in the radial metric is small — of order v^2/c^2 for a planet moving at orbital velocity v — but it is not zero. When this stretched radial distance is substituted into the geodesic equation, the centrifugal term L^2/r^2 picks up a multiplicative correction from the stretching factor:

$$L^2/r^2 \times (1 + r_s/r) = L^2/r^2 + L^2 r_s/r^3 = L^2/r^2 + 2GML^2/(c^2r^3)$$

The second piece is the relativistic correction, now appearing as an additional attractive term (note the sign: it adds to the inward pull, it does not resist it). Combined with the sign conventions of the effective potential:

$$V_{eff} = -GM/r + L^2/(2r^2) - GML^2/(c^2r^3)$$

The relativistic term is attractive and proportional to $1/r^3$. Newton's gravity is attractive and proportional to $1/r^2$. The centrifugal barrier is repulsive and proportional to $1/r^2$. The relativistic term tips the balance: at small r, it overwhelms the centrifugal barrier, pulls the orbit slightly inward, and prevents the orbit from closing. Each passage around the Sun the perihelion has advanced slightly because the planet dipped just a little deeper than a closed Newtonian ellipse would allow.

box: The $1/r^3$ term is the sole cause of Mercury's precession. It is present in the complex Quaternion metric. It is absent from Newton's dr. That one extra factor of $1/\sqrt{f}$ in the radial component — the spatial metric distortion — is the entire difference between a closed Newtonian ellipse and a precessing relativistic orbit. Forty-three arcseconds per century. One term. One geometric origin.

The third term is entirely absent in Newton — it comes from the $1/\sqrt{f}$ factor in the complex Quaternion radial component. It produces a small extra force proportional to $1/r^4$ that breaks the closure of the orbit. The perihelion advance per orbit is:

$$\Delta\varphi = 6\pi GM_\odot / (c^2 \times a \times (1-e^2))$$

where a (semi-major axis) and e (eccentricity) are Mercury's orbital parameters.

Numbers for Mercury

$$\Delta\varphi = 6\pi \times 1.327\times10^{20} / (9\times10^{16} \times 5.791\times10^{10} \times 0.9577) = 5.02\times10^{-7} \text{ rad/orbit}$$

$$415.2 \text{ orbits/century} \times 5.02\times10^{-7} \text{ rad} \times 206{,}265 \text{ arcsec/rad} = 42.98 \text{ arcsec/century}$$

Mercury precession — result
complex Quaternion metric predicts: 42.98 arcseconds per century
Observed (unexplained residual): 43.0 ± 0.5 arcseconds per century
Agreement to better than 1 percent.

Source: the $1/\sqrt{f}$ factor in the complex Quaternion radial component $dr/\sqrt{f}$.
Newton had no spatial metric. The complex Quaternion does.
That single difference explains a sixty-year astronomical mystery.

Both tests from one source

Light deflection and Mercury precession have the same algebraic origin: the spatial metric term $1/\sqrt{f}$ in the complex Quaternion radial component. For a photon ($ds^2 = 0$) this term doubles the deflection angle. For a massive planet ($ds^2 < 0$) it adds the $1/r^4$ force term that precesses the orbit. One term. Two famous experiments. One century of confirmation.

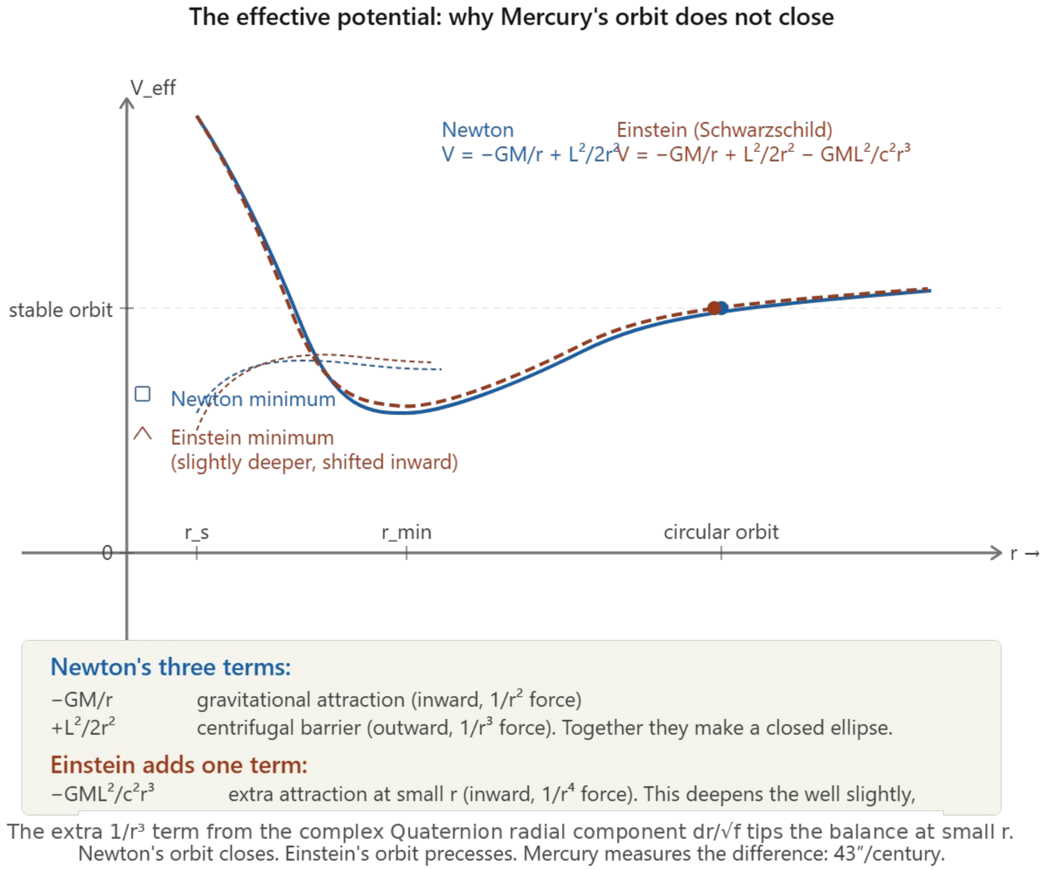


Figure 5. The effective potential. Newton's two terms alone close the orbit; the complex Quaternion adds $-GML^2/(c^2r^3)$, deepening the well at small r and shifting the minimum inward.

Chapter 6. The Exact Horizon Formula and the Collapse Threshold

6.1 The horizon as an exit cone

An observer stands on a sphere of radius R, eyes at height h above the surface. The horizon is the tangent line from the eyes to the sphere — the line that just grazes the surface. The tangent is always perpendicular to the radius at the point of contact. From the right triangle formed by eye, centre, and tangent point:

$$\cos\theta = R / (R + h)$$

The geometric relation is exact for a sphere viewed from height h, and it carries the right intuition: a horizon appears when some directions no longer lead out. For a mass M the exact statement does not need the identification $h = r_s$ to be assumed — it can be derived from the vacuum complex Quaternion metric itself. We do that now.

A photon in the vacuum metric $f(r) = 1 - r_s/r$ travels on the zero-norm surface of the complex Quaternion (Chapter 5). For a ray in the equatorial plane, two quantities are conserved: E, because the metric does not depend on time, and L, because it does not depend on the angle φ — the same two conservation laws that governed planetary orbits in Section 5.2. Their ratio defines the impact parameter $b = cL/E$: for a ray arriving from far away, b is the aiming distance of the ray from the centre. The zero-norm condition gives the radial equation:

$$(dr/d\lambda)^2 = (E/c)^2 \cdot [\, 1 - b^2 \cdot f(r)/r^2 \,] \qquad (\lambda = \text{parameter along the ray})$$

The ray turns around where the bracket vanishes: $b^2 \cdot f(r)/r^2 = 1$. Everything therefore depends on the single function $f(r)/r^2$. It has exactly one maximum: setting $d/dr[(1 - r_s/r)/r^2] = 0$ gives $-2/r^3 + 3r_s/r^4 = 0$, one line of algebra, and the solution is:

$$r = 3r_s/2 \qquad \text{(the photon sphere)}$$

At this radius — half again as far out as the horizon — light can orbit the mass in a circle. The maximum value of f/r^2 there is $(1/3)/(9r_s^2/4) = 4/(27 \cdot r_s^2)$, so the critical impact parameter is:

$$b_c = (3\sqrt{3}/2) \cdot r_s \approx 2.60 \cdot r_s$$

A photon aimed with $b < b_c$ crosses the photon sphere and is captured; one with $b > b_c$ escapes. Now place a static observer at radius r. The observer measures angles with proper lengths — radial distances stretched by $1/\sqrt{f}$, transverse distances $r \cdot d\varphi$ untouched: exactly the deformation of the complex Quaternion radial component that doubled the light deflection in Section 5.1. Working out the angle ψ (psi, Greek letter) between a light ray and the straight-out radial direction gives $\sin \psi = b \cdot \sqrt{f}/r$. Combining this with b_c: the directions that lead out form a cone around straight-out — the exit cone — whose half-angle is:

$$\sin \psi_{exit} = (3\sqrt{3}/2) \cdot (r_s/r) \cdot \sqrt{(1 - r_s/r)}$$

The exit cone at three radii — one formula
Far away ($r \gg r_s$): $\sin \psi \approx 2.60 \cdot r_s/r$. The mass shows as a dark disc of angular radius $2.6 \cdot r_s/r$ — the black-hole "shadow," 30 percent wider than the horizon itself.
The Event Horizon Telescope image of M87* (2019) [13] shows this disc, diameter $2b_c \approx 5.2 \cdot r_s$.
Photon sphere ($r = 3r_s/2$): $\sin \psi = 1$, $\psi = 90°$. The exit cone opens to a half-sky: a hovering observer sees half the view as sky, half as black.
Horizon ($r \to r_s$): $\sqrt{f} \to 0$, so $\psi_{exit} \to 0$. The exit cone closes completely. No direction leads out. This is the event horizon — derived, not asserted.

The closing of the exit cone at $r = r_s$ is the derived form of what the tangent-line picture suggested: the horizon closes when the object sits at its own Schwarzschild radius. The flat-space formula $\cos \theta = R/(R + h)$ remains a picture — useful, but not the calculation. The exit-cone formula is the calculation, and it comes from the same single function f(r) that produced the light deflection and the perihelion precession.

> **M = 0 (empty space, f = 1 everywhere):** No photon sphere, no capture — every direction leads out. No horizon.
>
> **R_object > r_s (any star; below the collapse threshold of Section 6.2):** The surface sits outside the horizon; the exit cone at the surface is open. Light escapes. Star visible.
>
> **R_object = r_s (black hole):** The exit cone at the surface has closed. Nothing escapes.

6.2 The collapse threshold from two Quaternion conditions

In Paper 3 of this series [7] we show that the Pauli exclusion principle is a theorem of Quaternion algebra: for any Quaternion Q representing a fermion state, the wedge product $Q \wedge Q = 0$ (wedge, pronounced "and", is the antisymmetric product $(QP - PQ)/2$ — it vanishes for identical Q by antisymmetry). This means no two neutrons can occupy the same momentum state. The resistance generates neutron degeneracy pressure that holds a neutron star up.

Maximum degeneracy pressure is set by the speed of light. At that limit the minimum neutron star radius for mass M is approximately $R_{NS}(M) \approx 1.2\times10^4\ (M_\odot/M)^{1/3}$ metres. Setting this equal to $r_s = 2GM/c^2 = 2.95\ (M/M_\odot)$ km and solving gives $(M/M_\odot)^{4/3} = 12{,}000/2{,}950 \approx 4.07$, i.e. $M \approx 2.9\ M_\odot$, with $R \approx 8.5$ km at the crossing. This is the collapse threshold: the mass above which no degeneracy pressure can hold the star outside its own horizon. It lies in the accepted range of the Tolman–Oppenheimer–Volkoff (TOV) limit for neutron stars, roughly 2.2–2.9 $M_\odot$. It is the mass at which the Quaternion Pauli condition $Q \wedge Q = 0$ and the complex Quaternion horizon condition $R_{object} = r_s$ are simultaneously satisfied. The historical ancestor of this degeneracy-limit reasoning is Chandrasekhar's 1931 limit of 1.4 $M_\odot$ [8]; that limit applies to white dwarfs supported by electron degeneracy, and the neutron-star threshold computed here is its analogue one collapse further on.

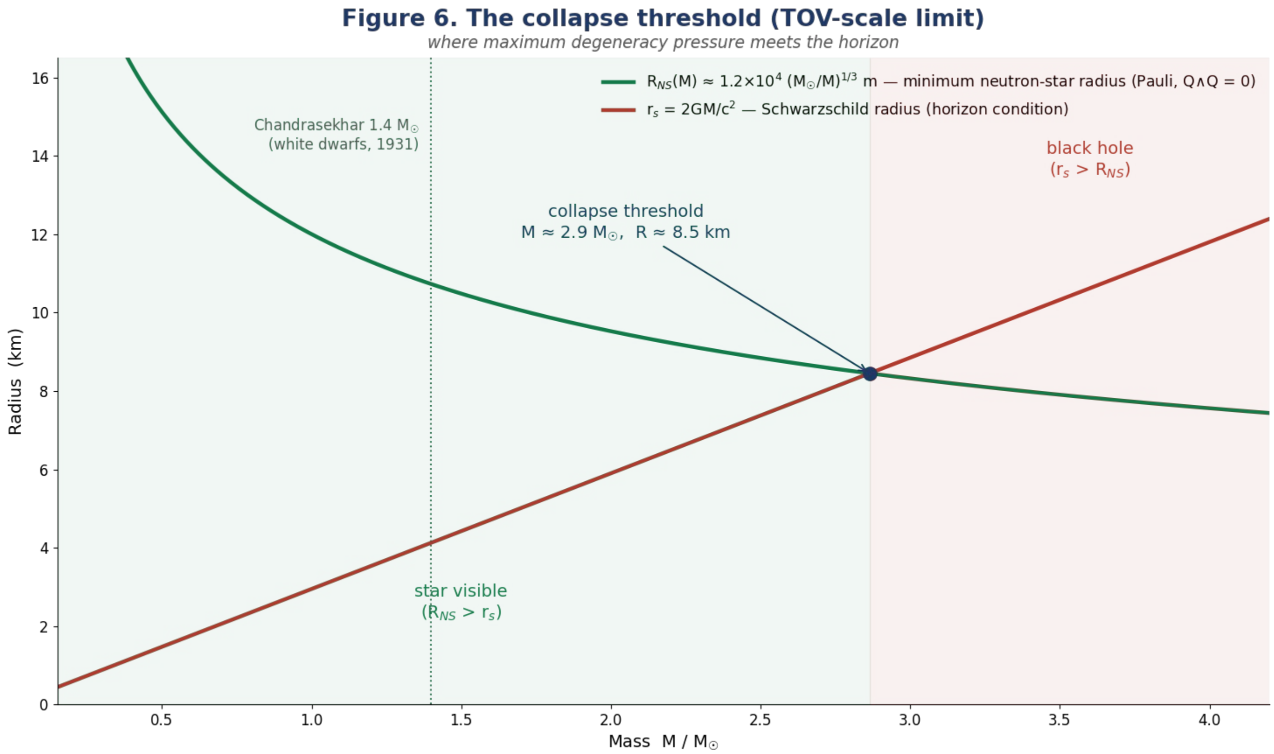


Figure 6. The collapse threshold (TOV-scale limit) as the intersection of two curves.
Green (descending): neutron star radius from Q∧Q = 0 (Pauli condition).
Red (ascending): Schwarzschild radius r_s = 2GM/c² (horizon condition).
The curves cross at ≈ 2.9 M☉, R ≈ 8.5 km.
Left: star visible. Right: black hole.
Both conditions come from Quaternion algebra. Their crossing is not a coincidence.

Chapter 7. Empty Space Has Curvature

In Chapter 4 we set G = 0 outside the mass and assumed f → 1 at large r. This assumed flat empty space. But is empty space flat?

7.1 The approximation we made and when it breaks

The vacuum condition G = 0 produces f(r) = 1 − r_s/r, which approaches 1 as r → ∞. This assumes that spacetime is flat (Minkowski) far from the mass. For solar system calculations — where r is at most a few light-hours and R = c/H is roughly 14 billion light-years (the 13.8-billion-year age of the universe is a different, coincidentally similar number) — this is an excellent approximation. The error is of order $r^2/R^2 \approx 2\times10^{-39}$ at Earth's surface. Negligible.

But the approximation breaks when we ask about the large-scale structure of spacetime. At cosmological distances r ~ R, the term we dropped becomes significant. And if it is significant, it must be present in the exact vacuum metric. The question is: what generates it?

7.2 The universe already has curvature

In Paper 2 of this series [6] we show that the universe is not expanding but infalling — a sphere of radius $R = c/H$, where H is the Hubble constant ($H \approx 2.27 \times 10^{-18}$ per second), with a gravitational metric that curves space everywhere. Empty space, in the complete absence of any local mass, already has a background curvature with characteristic scale R.

This is not an abstract statement. It has a direct consequence for the vacuum complex Quaternion. The boundary condition $f \to 1$ as $r \to \infty$ is wrong. The correct boundary condition is $f \to f_{background}$ as $r \to \infty$, where the background metric is:

$$f_{background}(r) = 1 - r^2/R^2$$

This is the de Sitter metric (Willem de Sitter, 1917 [5]), now identified from the infall cosmology of Paper 2: $R = c/H$ is the Hubble radius, not an abstract parameter. Even in the complete absence of any local mass, spacetime has this background curvature.

7.3 The full vacuum complex Quaternion metric

When a local mass M is present, its deformation is added to the background. The full vacuum condition — applied to the complex Quaternion in the cosmological background rather than in flat space — gives:

$$f(r) = 1 - r_s/r - r^2/R^2$$

This is our complete vacuum metric. It is not assumed. It is derived from the single requirement, now stated in its general form: outside the local mass, the curvature of the complex Quaternion reduces to the constant background curvature of the cosmological horizon. The local term r_s/r and the cosmological term r^2/R^2 emerge from the same derivation.

This result is known in tensor form as the Schwarzschild-de Sitter metric. Our contribution is: (1) $R = c/H$ is identified from the infall cosmology, not fitted as a free parameter; (2) both terms are derived from the single complex Quaternion vacuum condition; (3) the two horizons are identified as zeros of the same time component W; (4) the exit-cone analysis of Section 6.1 gives both horizons the same operational meaning: the closing of the directions that lead out.

One precision matters here. The Schwarzschild–de Sitter metric is not a solution of $G_{\mu\nu} = 0$: it solves $G_{\mu\nu} = -\Lambda g_{\mu\nu}$ with $\Lambda = 3/R^2$. The vacuum condition of Chapter 4 therefore generalizes from “the curvature vanishes” to “the curvature reduces to the constant background value” — in complex

Quaternion language, $d/dr[r \cdot f] = 1 - 3r^2/R^2$ (Section 4.5) in place of $d/dr[r \cdot f] = 1$. "Vacuum" in this chapter means free of local matter, not flat.

A further reconciliation is owed now that the series is complete [14]. The constant-curvature background in the form $f = 1 - r^2/R^2$ yields a redshift quadratic in distance at small d, and cannot by itself produce the exponential law $1 + z = e^{d/R}$ that Paper 2 establishes against observation. The operative background of the completed series is therefore the exponential line element constructed in the cosmology working note ("metric D"), with lapse $e^{-d/R}$ and matching transverse contraction; the Schwarzschild–de Sitter form of this chapter remains correct as the local approximation around a mass at $r \ll R$. The conceptual claims of this chapter survive intact — empty space is curved, the background scale is $R = c/H$, and the time component W vanishes at both extremes — with one honest refinement to Section 7.4: in the exponential background the outer zero of W is asymptotic rather than located at a finite radius. The two horizons are one exact zero and one approached forever.

7.4 Two horizons, one algebra

The full metric $f(r) = 1 - r_s/r - r^2/R^2$ has two zeros. Setting $f = 0$:

> **$r = r_s$ (inner horizon):** The black hole event horizon. $W = i\sqrt{f} \cdot c \cdot d\tau = 0$. Time component vanishes. Nothing escapes from inside this radius.
>
> **$r \approx R$ (outer horizon):** The cosmological horizon. $W = i\sqrt{f} \cdot c \cdot d\tau = 0$. Time component vanishes. No signal from beyond this radius can reach us.

Both are zeros of the time component W of the same complex Quaternion. The event horizon of a black hole and the edge of the observable universe are the same algebraic event — the vanishing of the imaginary time component — at two different scales separated by roughly twenty-two orders of magnitude: $r_s(\text{Sun}) \approx 3\times10^3$ m against $R \approx 1.32\times10^{26}$ m.

> **The full vacuum metric — one formula, all scales**
> $f(r) = 1 - r_s/r - r^2/R^2$
>
> Near a mass ($r \ll R$): $f \approx 1 - r_s/r$ (solar system, drop cosmological term)
> Far from mass ($r_s \ll r$): $f \approx 1 - r^2/R^2$ (cosmological scale, drop mass term)
>
> $r_s(\text{Earth}) = 8.87$ mm. $r_s(\text{Sun}) = 2.95$ km. $R = 1.32 \times 10^{26}$ m.
> At Earth's surface: $r^2/R^2 \approx 10^{-30} \times r_s/r$. Safely negligible.

7.5 Earth as a point mass

For any observer at distance r from Earth's centre with r greater than Earth's radius (6,371 km), Earth's gravitational field is identical to that of a point mass at r = 0. This follows from the complex Quaternion vacuum metric: outside any spherical mass distribution, the metric depends only on the total mass M and the distance r. The interior is invisible to the exterior geometry.

Earth's Schwarzschild radius is:

$$r_s(\text{Earth}) = 2GM_{\oplus}/c^2 = 2 \times 6.674\times10^{-11} \times 5.972\times10^{24} / (3\times10^8)^2 = 8.87 \text{ mm}$$

A marble. For a satellite at 400 km altitude (r = 6,771 km), the gravitational field is indistinguishable from what an 8.87 mm point mass would produce. The International Space Station does not know and does not care whether the object below it is a planet or a marble. Same M, same r, same f(r), same geodesics.

This is Birkhoff's theorem (George Birkhoff, 1923): the exterior vacuum complex Quaternion metric is the unique spherically symmetric solution, regardless of what happens inside. It follows directly from our derivation — we derived $f(r) = 1 - r_s/r$ using only the condition outside the mass and never needed to know what is inside.

7.6 Preview of Paper 2: the cosmological redshift

The background metric $f(r) = 1 - r^2/R^2$ of the cosmological infall gives a gravitational redshift for photons traveling across the universe. A photon emitted at distance d from us and traveling toward us climbs out of a gravitational well described by f(r). Integrating the metric over the photon's path gives:

$$1 + z = e^{d/R} = e^{Hd/c}$$

This is an exponential redshift formula. It reproduces the Hubble law $z \approx Hd/c = d/R$ at small d, and differs from the standard cosmological model (ΛCDM) at high redshift where $z \gtrsim 2$. The James Webb Space Telescope is observing galaxies at z = 10 to 16, exactly where the two models diverge. The derivation and observational predictions are developed in Paper 2 [6].

What Paper 2 derives from the background metric

$1 + z = e^{Hd/c}$ — exponential redshift from gravitational infall

At small z: $z \approx Hd/c$ (Hubble's law — agrees with all observations)
At large z: differs from ΛCDM (testable with JWST at z > 2)

> De Sitter's redshift goes as d^2 (quadratic) — not the same as ours.
> Standard ΛCDM uses a dynamical expanding metric — not the same as ours.
> Our formula is an exact derivation from $f(r) = 1 - r^2/R^2$ with $R = c/H$.

Chapter 8. The Real Part Passing Through Zero

A single algebraic event connects physics across thirty-six orders of magnitude: the real component of the complex Quaternion passes through zero.

At the atomic scale: during a quantum transition the electron moves between energy states. The state Quaternion $Q(t) = \cos(\omega t/2) + \sin(\omega t/2)\cdot j$ has a real part that passes through zero at the midpoint. The electron is neither in the initial nor the final state — it is purely Quaternion-imaginary, unobservable. We call this the transfiguration point (Paper 5 [9]).

At the stellar scale: the time component $W = i\sqrt{f}\cdot c\cdot d\tau$ vanishes when $f = 0$ at $r = r_s$. The complex Quaternion becomes purely Quaternion-imaginary. The event horizon is the zero of W.

At the cosmological scale: the metric time component vanishes at the boundary of the observable universe at $r = R$. The cosmological horizon is also the zero of W.

> **The real part reaches zero at every scale**
> Atom (10^{-10} m): state Quaternion real part = 0 at the quantum transition
> Black hole (km): $f = 0$ at $r = r_s$ at the event horizon
> Universe (10^{26} m): $f = 0$ at $r = R$ at the cosmological horizon
>
> Same algebra. Same event. Scales differ by a factor of 10^{36}.
> This is not a metaphor. It is the same equation at different values of r.

Chapter 9. Discussion

9.1 What has been shown

From a single object — the complex Quaternion $dQ = ic\cdot d\tau + dx\cdot\iota + dy\cdot j + dz\cdot k$ — we have derived from the single postulate $W = ic\cdot d\tau$:

The Lorentzian metric signature. The invariant spacetime interval. The Lorentz transformation. Time dilation and length contraction. $E = mc^2$. The vacuum metric $f(r) = 1 - r_s/r$. Einstein's field equations as the curvature condition. The 1919 light deflection of 1.75 arcseconds. The Mercury perihelion precession of 43 arcseconds per century. The event horizon as the zero of the time component. The neutron-star collapse threshold at approximately 2.9 solar masses. The full metric $f(r) = 1 - r_s/r - r^2/R^2$ including cosmological curvature.

The two-horizon structure. The exponential redshift formula (previewed; derived in Paper 2).

9.2 The unification argument

These results were previously derived from separate, unconnected mathematical frameworks: tensor calculus, Hilbert-space quantum mechanics, statistical mechanics, Friedmann cosmology. Each framework has its own axioms, its own language, and its own specialists. There was no obvious reason why they should be connected.

The complex Quaternion connects them. The Pauli exclusion principle and the neutron-star collapse threshold are connected by the intersection of $Q \wedge Q = 0$ and the horizon condition $R_{object} = r_s$. The quantum leap and the black hole event horizon are the same algebraic event at scales thirteen orders of magnitude apart. The Lorentzian signature of spacetime, which Einstein and Minkowski postulated, follows from $W^2 = (ic \cdot d\tau)^2 = -c^2 d\tau^2$. Poincaré's ict, introduced as a trick, is the correct identification of W. Hamilton's Quaternion, carved into a bridge in 1843, contains the structure of spacetime.

A single coincidence is possible. Two coincidences are suspicious. Eight coincidences across completely different areas of physics, all following from the same algebraic structure, are not coincidences. They are the structure revealing itself.

9.3 What is genuinely new

We are clear about what we have derived and what we have borrowed. The Lorentzian signature, special relativity, the Schwarzschild metric, and the Schwarzschild-de Sitter metric were known before us. We did not discover them. We derived them from a more fundamental starting point.

What is genuinely new: the identification $W = ic \cdot d\tau$ as the single postulate from which all the above follow — within this paper's scope; the completed series rests on three postulates, of which this is the first, stated together in the foundations paper [14]. The two-wall proof that the imaginary character of time is permanent. The presentation of the horizon as the closing of the photon exit cone, computed within the complex Quaternion framework and anchored by the elementary tangent-line picture (the exit-cone result itself is standard general relativity). The identification of $R = c/H$ from the infall cosmology rather than from a fitted parameter. The identification of the event horizon and the cosmological horizon as zeros of the same W. The exponential redshift formula (Paper 2). The identification of the neutron-star collapse threshold as the intersection of two Quaternion conditions. The programme of deriving Pauli exclusion, quark confinement, and neutron decay from the same algebraic framework (Papers 3, 4, 5).

9.4 The new prediction

The exponential redshift formula $1 + z = e^{Hd/c}$ differs from the standard cosmological model (ΛCDM) at redshifts $z \gtrsim 2$. At $z = 10$, the two models predict different luminosity distances, different angular sizes, and different apparent brightnesses for galaxies. The James Webb Space Telescope is observing at $z = 10$ to 16. Early results show galaxies that are unexpectedly large and bright at high redshift — a tension with ΛCDM that is consistent with the exponential formula. This is not yet a confirmed refutation of ΛCDM, but it is exactly the observation that can distinguish the models. Paper 2 develops the quantitative predictions in detail.

Chapter 10. Conclusion

Hamilton carved $i^2 = j^2 = k^2 = ijk = -1$ into a stone bridge in 1843. Poincaré wrote $l = ict$ in 1905. Minkowski built the geometry of spacetime in 1908. Einstein found the curvature equations in 1915. Schwarzschild solved them in 1916. De Sitter found the cosmological metric in 1917. They were all reaching toward the same object. None of them called it a complex Quaternion, but each of them found one piece of it.

The complex Quaternion $dQ = ic\cdot d\tau + dx\cdot\imath + dy\cdot j + dz\cdot k$ is that object. Its norm is the spacetime interval. Its rotation group is the Lorentz group. Its curvature condition in vacuum is Einstein's field equation. Its two zeros are the black hole horizon and the cosmological horizon. Its Newtonian limit gives $E = mc^2$. Its algebraic properties give the Pauli exclusion principle. Its full vacuum metric $f(r) = 1 - r_s/r - r^2/R^2$ unifies local gravity and the geometry of the cosmos in one formula.

Empty space is not flat. It carries a background curvature with radius $R = c/H$. Mass adds a local curvature on top of it. The two curvatures together produce two horizons — one small, one vast — that are the same algebraic event in the same complex Quaternion. The universe, from the quantum leap to the edge of the observable cosmos, is described by one object and one condition: the real part of the complex Quaternion passes through zero at the boundary of the observable.

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Appendix. Greek Letters and Mathematical Operators

Every non-standard symbol listed on first appearance, with name and usage.

Greek letters
ι iota — first Quaternion imaginary axis (spatial x-direction)
τ tau — real time coordinate ($W = ic \cdot d\tau$)
γ gamma — Lorentz factor $1/\sqrt{(1-v^2/c^2)}$, always ≥ 1
φ phi — rapidity of a Lorentz boost; also azimuthal angle
θ theta — polar angle; also the horizon angle
λ lambda — summation index in tensor expressions; also the parameter along a light ray (6.1)
μ mu — tensor index label (one of the four spacetime directions)
ν nu — second tensor index label
ρ rho — three-momentum magnitude; also mass density
ω omega — angular frequency of a quantum transition
π pi — 3.14159… (appears in $8\pi G/c^4$)
Γ Gamma — Christoffel symbol, gradient of the geometry

Λ Lambda — cosmological constant ($\Lambda = 3/R^2$ in standard notation)
ψ psi — half-angle of the photon exit cone (Section 6.1)

Mathematical operators
∂ partial — rate of change in one direction, others fixed
$\wedge$ wedge — antisymmetric product $Q \wedge P = (QP-PQ)/2$
$Q \wedge Q = 0$ for identical Q (Pauli exclusion)
d/dr d by dr — ordinary derivative with respect to r
√ root — square root
≡ triple equals — defined as equal to (not merely happens to equal)
≪ much less — $v \ll c$ means v is negligibly small compared to c
≫ much greater — $R \gg r_s$ means R is enormously larger than r_s
tanh tanh — hyperbolic tangent, $\tanh(\varphi) = (e^{\varphi}-e^{-\varphi})/(e^{\varphi}+e^{-\varphi})$
cosh co-shine — hyperbolic cosine, $\cosh(\varphi) = (e^{\varphi}+e^{-\varphi})/2$
sinh shine — hyperbolic sine, $\sinh(\varphi) = (e^{\varphi}-e^{-\varphi})/2$
$\cosh^2\varphi - \sinh^2\varphi = 1$ (hyperbolic identity, analogous to $\cos^2+\sin^2=1$)

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[i] Hendrik Antoon Lorentz was a Dutch physicist, born 1853 in Arnhem, died 1928. He was one of the most respected physicists of his era — Einstein called him the most important influence on his thinking, and when Lorentz walked into a room, Einstein would stand up. He won the Nobel Prize in 1902 with Pieter Zeeman for the discovery that magnetic fields split spectral lines, which was the first experimental hint that electrons exist inside atoms.

Lorentz spent most of his career trying to understand how light travels through space. The dominant theory of his time said light was a wave in a medium called the ether — a substance filling all of space, like water filling a lake. Waves need a medium. Sound needs air. Light, being a wave, must need something. So everyone assumed the ether existed.

The problem was that nobody could detect it. In 1887, Michelson and Morley built the most precise interferometer ever made and looked for the difference in the speed of light going with the Earth's motion through the ether versus against it. They found nothing. The speed of light was the same in both directions. This was deeply embarrassing for the ether theory.

Lorentz's response was brilliant and almost right. He proposed in 1895 that moving objects physically contract in the direction of their motion through the ether. A ruler moving through the ether gets shorter. This contraction would be exactly the right amount to cancel the Michelson-Morley result. The formula for how much shorter is what we now call the Lorentz factor:

$$\gamma = 1 / \sqrt{(1 - v^2/c^2)}$$

A ruler moving at velocity v is contracted to $1/\gamma$ of its rest length. He also proposed that moving clocks run slow by the same factor — what he called local time. These were the Lorentz transformations, derived in 1904.

The crucial thing is why Lorentz got the right formulas for the wrong reason. He thought the contraction was a real physical effect caused by the ether pressing on moving objects. He never abandoned the ether. He spent the rest of his life trying to reconcile his transformations with the idea that the ether was real.

Einstein, one year later in 1905, derived exactly the same transformations from two postulates — no ether required. The speed of light is the same for all observers. The laws of physics are the same for all observers. From those two statements the Lorentz transformations follow mathematically. Einstein never needed the ether and never used it.

So the transformations carry Lorentz's name, but the interpretation belongs to Einstein. Lorentz found the right equations. Einstein found out what they meant.

The interval

The Lorentzian interval is not Lorentz's invention. This is a common confusion.

Lorentz had his transformations. Einstein had his relativity. But neither of them first wrote down the geometric object we call the spacetime interval. That was Hermann Minkowski in 1908.

Minkowski was Einstein's mathematics professor in Zürich and had not been particularly impressed with the young Einstein as a student. When he read Einstein's 1905 paper on special relativity, he recognised something that Einstein had not fully grasped: the transformations were telling you that space and time were not two separate things but two aspects of a single four-dimensional geometry. He gave a famous lecture in Cologne in 1908 that began:

"Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

Minkowski then wrote down the object that measures distances in this four-dimensional geometry. In ordinary three-dimensional space, the distance between two points is:

$d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$

All positive. Always positive. This is Pythagoras.

In four-dimensional spacetime, Minkowski wrote:

$ds^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$

The minus sign. That is the whole story. The minus sign is what makes time different from space. It is what makes the spacetime interval sometimes positive, sometimes negative, sometimes zero — whereas ordinary distance is always positive.