Pauli Exclusion in the Quaternion Framework The Double Cover Gives the Sign; the Tensor Product Gives the Rule

Martin Scholl — Independent Researcher

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Abstract

The Pauli exclusion principle has exactly two ingredients, and this note assigns each to its rightful owner. There is a **sign** — the fact that a two-fermion state changes sign when the particles are exchanged — and there is a **rule** — that no two fermions may occupy the same state, which is the consequence of that sign. This series can claim the *sign* as genuinely quaternionic, and should claim nothing more. Fermions are half-windings of the quaternion sphere: the rotation group of space is doubly covered by the unit quaternions SU(2), in which a 4π turn is the identity and a 2π turn is -1, a fact measured directly in neutron interferometry in 1975. By the Finkelstein–Rubinstein argument, exchanging two identical such objects is equivalent to rotating one of them by 2π — and so multiplies the joint state by -1. That minus sign is the whole quaternionic dividend, and it is a real one. The *rule* is then standard quantum mechanics, and this note states it correctly: exclusion lives in the tensor product of single-particle Hilbert spaces, where the two-fermion state is the antisymmetrised (Slater) combination $|\alpha\rangle \wedge |\beta\rangle$, whose squared norm $1 - |\langle\alpha|\beta\rangle|^2$ vanishes **if and only if** the two single-particle states are the same ray, and is nonzero for *every* distinct pair. The quaternion earns a third, humbler role: it is an excellent **label** for a single electron's quantum numbers, packing energy, charge, spin, and a complex angular momentum onto its four axes — a legible identity card for $|\alpha\rangle$. What it does not do, and what an earlier draft of this series wrongly asked of it, is *perform* the antisymmetrisation: a binary product of two four-component labels — in particular the quaternion commutator $\frac{1}{2}(QP - PQ)$, which is merely the cross product of the imaginary parts and is blind to the energy axis — cannot reproduce the exclusion rule, because the rule depends on the inner product $\langle\alpha|\beta\rangle$ of the *full* states, which no product of labels computes. That correction is made here once and then left behind; the construction below is built correctly from the start. The honest summary: the double cover supplies the exchange sign, the Hilbert-space tensor product turns that sign into the exclusion rule, and the quaternion names the states — three clean roles, one of them genuinely the algebra's. A standard-library script, pauli_check.py, confirms the construction gives the right verdict on every pair, including the shells (1s, 2s) that a label-commutator fails. Every symbol is introduced before use; the deep 'why half-integer spin?' — the spin–statistics theorem — is named as the standing debt it is.

1. Two Ingredients, Two Owners

The exclusion principle forbids two identical fermions from sharing a quantum state, and from that single rule flow the periodic table, the stability of matter, and the degeneracy pressure of white dwarfs. It is usually stated as a postulate. Underneath the postulate are two separable facts. First, a **sign**: the joint wavefunction of two identical fermions is antisymmetric under their exchange, $\Psi(1,2) = -\Psi(2,1)$. Second, a **rule** that follows from the sign: set the two particles in the same state and $\Psi(1,1) = -\Psi(1,1) = 0$ — the amplitude vanishes, so the configuration is forbidden. To 'explain' Pauli is to explain the sign; the rule is then automatic. This note keeps the two apart, because their origins are different: the sign is where the quaternion genuinely enters, and the rule is standard Hilbert-space bookkeeping.

2. The Quaternion as a State Label

Give each single-particle state a legible name. This series encodes an electron's quantum numbers on the four axes of a **state quaternion**:

$$Q = (E/\Lambda)\cdot e_0 + q\cdot e_1 + m_s\cdot e_2 + (\ell + i\, m_\ell)\cdot e_3,$$

with the energy on the real axis e_0 (Λ = 310 MeV the confinement scale), the electric charge on e_1, the spin projection on e_2, and — a genuinely useful device — the orbital angular momentum on the complex axis e_3, carrying ℓ in its real part and m_ℓ in its imaginary part, so that the three 2p states ($m_\ell = -1, 0, +1$) are held apart as $1-i$, 1, $1+i$. This is a compact, faithful identity card for the state we will call $|\alpha\rangle$, $\alpha = (n, \ell, m_\ell, m_s)$. It is a label, and a good one; the physics of exclusion does not happen *in* it, but it names the objects to which the physics applies.

3. The Double Cover Supplies the Exchange Sign

Here is the quaternion's real contribution. The rotations of ordinary space form the group SO(3); the unit quaternions form SU(2), which covers SO(3) **twice** — every spatial rotation corresponds to *two* quaternions, q and −q, and a continuous 2π rotation returns the quaternion not to +1 but to −1, with only a 4π turn restoring the identity. A spin-½ particle is an object that lives on this double cover: it is a *half-winding* of the quaternion sphere. This is not metaphor — it was measured, in 1975, by passing neutrons through a magnetic field and observing that a 2π rotation flips the sign of the beam's interference, while 4π restores it.

Now the exchange. By the Finkelstein–Rubinstein theorem, for solitonic objects the operation of **exchanging two identical particles is topologically equivalent to rotating one of them by 2π** while the other is held fixed. For a double-cover object, that 2π rotation multiplies the state by −1. Therefore:

$$\textit{exchange of two identical spin-½ windings} \implies \times(-1).$$

That is the antisymmetry sign, and it is delivered by the quaternion double cover — the one piece of Pauli that this framework can honestly call its own. (What the double cover does *not* by itself prove is that half-integer spin *must* pair with this sign for *all* fields — that is the spin–statistics theorem of relativistic quantum field theory, §7. The double cover shows *how* the sign arises for a winding; the theorem shows *why* it is compulsory.)

4. The Tensor Product Turns the Sign into the Rule

With the sign in hand, exclusion is the standard antisymmetry of the multi-particle Hilbert space. Single-particle states are orthonormal vectors $|\alpha\rangle$ in a Hilbert space $\mathcal{H}$; a two-fermion state is the antisymmetrised combination — the Slater determinant, an element of the exterior square $\Lambda^2\mathcal{H}$:

$$|\alpha\rangle \wedge |\beta\rangle = (|\alpha\rangle \otimes |\beta\rangle - |\beta\rangle \otimes |\alpha\rangle)/\sqrt{2}.$$

The minus sign is exactly the exchange sign of §3, now doing its work. The squared norm is a clean, universal formula,

$$\| \, |\alpha\rangle \wedge |\beta\rangle \, \|^2 = 1 - |\langle\alpha|\beta\rangle|^2,$$

which vanishes **iff** $|\alpha\rangle$ and $|\beta\rangle$ are the same ray and is nonzero for every distinct pair. This is correct for *all* states — crucially including two states that differ only in energy, such as 1s↑ and 2s↑, which are orthogonal ($\langle 1s|2s\rangle = 0$) and therefore give $\|\alpha\wedge\beta\|^2 = 1$, freely allowed, exactly as lithium requires:

Pair	**⟨α|β⟩**	**‖α∧β‖²**	**verdict**
1s↑ , 2s↑ (differ in n)	0	1	allowed — lithium's third electron
1s↑ , 1s↑ (same ray)	1	0	forbidden
1s↑ , 1s↓ (differ in spin)	0	1	allowed — helium ground state
2p m=−1 , 2p m=+1 (differ in m_ℓ)	0	1	allowed

The operation is a genuine wedge — the antisymmetric ∧ the principle demands — but it is the exterior product of the *Hilbert space*, acting on state *vectors* through their inner product, and the quaternion of §2 is the label of the vectors, not the operator. It is worth saying once why a label-product cannot stand in for it: any bilinear product of two four-component labels — the quaternion commutator ½(QP − PQ) included — is a function of the labels alone; but $\|\alpha\wedge\beta\|^2$ depends on $\langle\alpha|\beta\rangle$, the overlap of the *full* states, which no product of labels reconstructs. (The commutator specifically equals Im(Q)×Im(P), the cross product of the imaginary parts: it never sees the real/energy axis and vanishes whenever the imaginary parts are parallel, so it would wrongly forbid 1s↑ and 2s↑. pauli_check.py prints that zero, and confirms the tensor-product rule above gets every pair right.)

5. Filling the Table

The two pieces together generate atomic structure with no further postulate. Each shell is a finite set of orthonormal labels $|\alpha\rangle$; the exclusion rule permits any antisymmetrised combination and forbids repetition of a ray; so a shell of k distinct states holds exactly k electrons, and the (k+1)-th must open the next shell. The n = 2 shell has eight labels (2s and three 2p, each with two spins), all mutually orthogonal, hence all mutually allowed and jointly filled; a ninth electron would have to duplicate a ray, giving $\|\alpha\wedge\alpha\|^2 = 0$. The periodic table's period lengths 2, 8, 8, 18, … are the counts of orthonormal labels per shell — the quaternion labels them, the tensor product excludes duplicates, and chemistry follows.

6. Why a Single Quaternion Cannot Carry the Rule — Stated Once

For the record, and so the design choice is not repeated: exclusion cannot be a binary operation on two state labels producing a number. The reason is structural, not a matter of choosing a better product. The rule must vanish for identical states and *only* for identical states, i.e. it must be a function of the overlap $\langle\alpha|\beta\rangle$ of the full Hilbert-space vectors. A product of four-component labels is a function of the labels, and the label is a lossy summary — two orthogonal states (1s, 2s) can carry parallel imaginary parts, and any product blind to the real axis (the commutator is exactly such) collapses them. The antisymmetrisation therefore *must* be performed in the Hilbert space, where the inner product lives; the quaternion's job ends at naming the vectors.

7. What Is Quaternionic, What Is Standard, What Is Owed

Quaternionic (claimed). The exchange *sign* −1, from the double cover SU(2) and the Finkelstein–Rubinstein exchange-equals-rotation argument — a genuine, measured (1975) contribution of Hamilton's algebra to the foundations of matter. The state *label* — a faithful, legible encoding of the quantum numbers.

Standard (used, not claimed). The exclusion *rule* itself: the tensor-product antisymmetrisation, $\|\alpha\wedge\beta\|^2 = 1 - |\langle\alpha|\beta\rangle|^2$. This is ordinary quantum mechanics; the framework adopts it and supplies its sign.

Owed. The deep 'why' — that half-integer spin *must* carry the antisymmetric sign for every field — is the **spin–statistics theorem** (Fierz 1939; Pauli 1940), a result of relativistic quantum field theory. The double cover shows how the sign arises for a winding and is fully consistent with the theorem; it does not replace it. This series may say: Pauli's sign is the double cover's; Pauli's rule is the tensor product's; Pauli's necessity is spin–statistics'. Nothing is hidden in the word 'consequence'.

8. The Sentence

Exclusion is a sign and a rule with two owners: the quaternion double cover, half-wound and measured in 1975, hands over the minus sign of exchange; the Hilbert-space tensor product turns that minus into the law that no two electrons may share a ray; and the quaternion, having given the sign, spends the rest of its service merely as the name on each electron's identity card.

References

W. Pauli, Z. Phys. 31, 765 (1925); M. Fierz, Helv. Phys. Acta 12, 3 (1939) and W. Pauli, Phys. Rev. 58, 716 (1940) — spin–statistics; D. Finkelstein and J. Rubinstein, J. Math. Phys. 9, 1762 (1968) — exchange as rotation; H. Rauch et al., Phys. Lett. A 54, 425 (1975) — the 4π neutron interferometry; standard treatments of the Slater determinant and the exterior algebra of Hilbert space; and the notes of this series (the State Quaternion label; the Three Storeys — the seal and the facade). Verification script: pauli_check.py (stdlib). This edition supersedes the earlier draft that defined exclusion via the quaternion commutator. (Citations from memory; the literature-verification pass applies.)

Acknowledgment: drafting and numerical checks by machine (Claude, Anthropic), in conversation; the instinct that exclusion should wear an antisymmetric wedge and that fermion statistics is the double cover's business is the author's — here placed on its correct footing, the sign from the algebra, the rule from the tensor product.