The Frame Quaternion: Field Equations in Hamilton's Variables

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Abstract. Einstein wrote gravity as ten coupled equations on a tensor. This note proposes — and partially executes — the same physics in this series' native variables: quaternions. The payoff is not cosmetic. In Hamilton's variables the wave operator splits into two first-order pieces (the trick by which Dirac beat his predecessors, here a property of the arena itself); all four of Maxwell's equations become one line; Einstein's equation becomes a first-order pair in which torsion — the twist that the tensor formulation amputates by decree — is simply present; and one template emerges across the rungs: *curvature of the winding equals source.* The programme's first steps are executed and verified symbolically: the new equations reproduce, exactly, everything the old formulation knew — Schwarzschild, metric D, all passed audits — and one open question of the series is decided by computation (galactic torsion fails by ninety-six orders of magnitude; the screw must live in the propagating fiber). The engine — deriving the pair from thermodynamics on the seals — remains the mountain, named. Every variable is defined at first use; a glossary closes the note.

1. The Arena and Its Derivative

All fields below are complex quaternions: objects $Q = q_0 + q_1 \cdot \mathbf{i} + q_2 \cdot \mathbf{j} + q_3 \cdot \mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are Hamilton's units (each squares to −1; $\mathbf{i} \cdot \mathbf{j} = \mathbf{k}$ and cyclically; reversed order flips the sign) and each coefficient may itself be complex through the *commuting* imaginary i — the same two-imaginaries structure as Paper 1, where time enters as $W = i \cdot \tau$. The conjugate $\bar{Q}$ flips the three Hamilton units; the norm $Q \cdot \bar{Q}$ is, for complex coefficients, not positive-definite — and that is precisely what produces the minus sign of spacetime's signature, Paper 1's founding theorem.

Now the one operator everything below uses. Package the four rates of change of spacetime into a single quaternion derivative:

$$D = -(i/c) \cdot \partial/\partial t + \mathbf{i} \cdot \partial/\partial x + \mathbf{j} \cdot \partial/\partial y + \mathbf{k} \cdot \partial/\partial z$$

Read it slowly: $\partial/\partial t$ is the time-rate of whatever D acts on, and the factor $-i/c$ places it on the imaginary time axis ($W = i\tau$ made operational) while converting units; the three spatial slopes ride the three Hamilton units. The central identity — checkable by direct multiplication with Hamilton's table — is that D times its conjugate gives the classical wave operator:

$$D \cdot \bar{D} = (\text{spatial curvature of the field}) - (1/c^2) \cdot (\text{its acceleration in time}) = -\Box$$

The second-order operator that governs all waves *factorizes* into two first-order quaternion operators. This is the same mathematical trick by which Dirac's first-order equation beat the second-order attempts before it — but here it is not a trick; it is a property of the arena. First-order field equations are what the algebra offers that tensor calculus cannot.

2. The Exemplar: All of Maxwell in One Line

Pack the electric field E (volts per metre) and magnetic field B (tesla) into one object, with the commuting i keeping them on separate layers:

$F = E + i\cdot c\cdot B$ (each field riding the Hamilton units: $E = E_x\cdot\mathbf{i} + E_y\cdot\mathbf{j} + E_z\cdot\mathbf{k}$, likewise B)

Pack charge density ρ and current density J into a source quaternion $\mathcal{J}$ the same way. Then **all four of Maxwell's equations are the four graded parts of the single statement**

$D\cdot F = -\mathcal{J}$

— one operator, one field, one source, one line. (Expanding the product with Hamilton's table, the dot-product parts of the multiplication deliver the two divergence laws and the cross-product parts deliver the two curl laws; the correspondence goes back to Silberstein, 1907.) Conservation of charge follows by hitting the equation once more with $\bar{D}$. This one-line standard is what the gravity equation must now meet.

3. The Frame Quaternion: the Variable That Replaces the Tensor

Why must the tensor go? Because matter, in this series, is spinorial (Postulate P1), and the Dirac equation cannot be coupled to a metric tensor at all — every textbook that puts spinning matter in gravity is forced to introduce the *frame field*: at each point, the local clock and the three local rulers. This series has carried that variable since Paper 1 without naming it. Name it now:

$\theta = \theta^0\cdot(i) + \theta^1\cdot\mathbf{i} + \theta^2\cdot\mathbf{j} + \theta^3\cdot\mathbf{k}$

θ is the **frame quaternion**. Its four coefficients are 1-forms — machines that eat a small displacement and return a number: θ^0 returns what the *local clock* reads for that displacement (carried on the imaginary axis, as time must be); θ^1, θ^2, θ^3 return what the three *local rulers* read. For the series' background, metric D:

$\theta = i\cdot c\cdot e^{(-kr)}\cdot dt + dr\cdot\mathbf{i} + e^{(-kr)}\cdot r\cdot d\vartheta\cdot\mathbf{j} + e^{(-kr)}\cdot r\cdot \sin\vartheta\cdot d\varphi\cdot\mathbf{k}$

(k = H/c the curvature per metre; r the proper distance; ϑ, φ the angles.) The metric — the ds^2 of every relativity course — is demoted to a byproduct: it is simply the norm of the frame, $ds^2 = \theta\cdot\bar{\theta}$, with the minus sign of time emerging from $i^2 = -1$ exactly as in Paper 1, now field by field.

4. The Connection Quaternion: Where the Screw Lives

One more object. As you move a small step, the local frame *tilts* — the clock and rulers at the new point are slightly rotated and boosted relative to the old. The bookkeeper of that tilting is the **connection quaternion** ω: a quaternion-valued 1-form whose pure-Hamilton part records spatial rotations and whose i-carrying part records boosts. (That rotations-plus-boosts fit exactly into a complex quaternion is no accident: the Lorentz algebra *is* the complex quaternions — one more structure the algebra supplies free.) Physically, ω is the gravitational field in first-order language. Algebraically, it is the object the dark-matter paper's screw twists.

5. The Structure Equations and the Proposed Pair

Two classical definitions (Cartan's), stated in words and packaged in quaternions. The **torsion** T measures the failure of small parallelograms to close: march along ruler one then ruler two, versus two then one, and T is the gap. The **curvature** R measures the rotation your frame has silently picked up after being carried around a small loop. In symbols: $T = d\theta + \omega \wedge \theta$ and $R = d\omega + \omega \wedge \omega$, where d is the boundary-measuring derivative and ∧ the antisymmetric product from the Pauli paper.

Note what the tensor formulation does at exactly this point: it *decrees* T = 0 — "the connection shall be symmetric" — and thereby amputates the antisymmetric sector of geometry before physics is even consulted. In Hamilton's variables torsion is simply present, and the screw acquires its geometric home.

The proposed field equations — replacing Einstein's ten second-order components with a first-order pair:

(I) $\langle R \wedge \theta \rangle = \kappa \cdot \mathcal{P}$ ($\kappa = 8\pi G/c^4$) (II) $T = \kappa_s \cdot \sigma$ ($\kappa_s = 8\pi G/c^3$)

In words: *(I)* curvature, paired with the frame, equals energy-momentum — this is Einstein's content in first-order clothing (the trade knows it as the tetrad-Palatini form); $\mathcal{P}$ is built from Paper 1's own momentum quaternion. *(II)* torsion equals spin density — the Einstein–Cartan relation: torsion does not propagate in empty space but switches on inside spinning matter; σ aggregates the state quaternions' spin content. Two flags, raised at once: the galactic screw needs torsion-like effects to *leak beyond* matter, which equation (II) as written does not provide (resolved by computation in Section 8a); and the vacuum limit must reproduce every audit the series has passed (verified in Section 8a).

6. The Ashtekar Alignment

None of this is an untrodden road, and the crossing of paths is remarkable. Ashtekar (1986) reformulated general relativity with an SU(2) connection as the fundamental variable — and SU(2) *is* the unit quaternions. In his self-dual variables the equations of gravity simplify dramatically and handedness is native. Mainstream physics has always lodged one complaint: the variables are complex, and awkward "reality conditions" must be bolted on to recover ordinary gravity. Read that complaint against this series' founding axiom. The formulation of gravity in which quaternions are natural is precisely the one whose supposed defect is that it wants time to be imaginary — and $W = i\tau$ is Postulate 1. We do not propose an exotic alternative to Einstein; we claim the self-dual formulation as the *native* one, with its reality condition promoted from embarrassment to physics.

7. One Template, Four Sectors

sector	connection (the potential)	curvature (the field)	equation	source (state-quaternion axis)
gravity	ω on the frame	$R = d\omega + \omega\wedge\omega$	$\langle R\wedge\theta\rangle = \kappa\mathcal{P}$	energy (e_0)
electromagnetism	A on the circle winding	$F = dA$	$D F = -\mathcal{J}$	charge (e_1)
weak	connection on the S^3 fiber	its curvature	gauged inside ω's fiber extension (Matter Meets Space)	spin/isospin (e_2)
matter	—	—	Lanczos–Conway first-order pair	the state quaternion itself

One template — **curvature of the winding equals source** — and one operator, D, in every row. Three structural corollaries fall out. *Linearity from commutativity:* the circle commutes with itself, so the $A\wedge A$ self-interaction term vanishes and electromagnetism is linear — light passes through light, superposition holds, the atom is exactly solvable. The computability of chemistry is the commutativity of a circle. The non-commuting rungs self-interact — which is why gravity gravitates and why the colour force anti-screens (the asymptotic-freedom note). *Coulomb is the Schwarzschild of its rung:* each equation's unique one-source spherical solution, with hydrogen as that rung's solar system. *The source ledger:* which state-quaternion axis feeds which equation is exactly the ledger of the State Octonion paper — energy feeds the frame, charge feeds the circle, spin feeds the sphere. This alignment was not designed; it emerged, which is the kind of evidence a framework cannot fake. What the electromagnetic rung still owes is not its law (g−2 agrees with $D F = -\mathcal{J}$ through twelve digits — the best-verified statement in physics) but its *gearbox*: α and its running.

8. The Programme, and What Has Been Executed

(i), (ii) — verified. The full first-order chain was run symbolically (companion script): frame θ → connection ω solved from $T = 0$ → curvature R → Einstein content. For Schwarzschild the result is exactly zero (vacuum, as it must be); for metric D the result reproduces term-by-term the tension-medium source already computed in the Four

Calculations note. The new equations know everything the old ones knew — the correct result for a change of variables, and the necessary one: every audit of the series survives the move to Hamilton's variables untouched.

(iii) — decided by computation. Can equation (II)'s algebraic torsion carry the galactic screw? No, twice over. By polarization: torsion couples to *net* intrinsic spin, and galactic matter's random spins cancel. By magnitude: even a fully spin-polarized hydrogen disk yields a torsion energy density of $\sim 10^{-101}$ joules per cubic metre against the $\sim 10^{-4}$ a halo-scale effect requires — **a shortfall of ninety-six orders of magnitude.** The decision is forced and final: the galaxy-scale screw lives in the propagating fiber connection, not in algebraic torsion. Torsion keeps its kingdom where the numbers put it: densities above $\sim 10^{76}$ kg/m^3 — deep black-hole interiors, where each collapse may seed its own inside. The theater's trapdoors, once more.

(iv) — open, and named. The engine: derive the pair (I)–(II) from the Clausius relation $\delta Q = T \cdot dS$ applied on the seals — Jacobson's 1995 construction, rerun in frame variables, so that the field equations become a *theorem* of the family law rather than a postulate. Four steps stand named: local horizon seals at every point; the family law supplying T; the entropy functional on the frame; the focusing step in first-order form. Until (iv) is done, the pair has the status this series knows how to name: a candidate, stated in advance, wearing its debts in public.

9. Glossary

symbol
i
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$
D
F, $\mathcal{J}$
θ
ω
T, R
$\mathcal{P}$, σ
κ, κ_s

References

L. Silberstein, Ann. Phys. 22, 579 (1907); É. Cartan (1923–25); C. Lanczos, Z. Phys. 57, 447 (1929); A. W. Conway, Proc. Roy. Soc. A 162, 145 (1937); A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986); T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995); N. Popławski (torsion bounce); and the papers and notes of this series (Paper 1; Matter Meets Space; the Four Calculations note; the State Octonion paper; verification: frame_quaternion_verification.py). *(Citations from memory; the literature-verification pass applies.)*